Relativistic approach of the LISA mission

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Abstract. The three LISA spacecraft aim at the interferometric detection of gravitational waves in the $[10^{-4}, 10^{-1}]$ Hz band. They are to be launched in 2014, 5 million kilometers appart, in a triangular configuration, orbiting around the Sun. This work in progress deals with the development of a simulator for the LISA mission (LISACode). It requires a description of orbitography and optical links in a native relativistic framework. The corresponding time delays, crucial for the TDI (Time Delay Interferometry), are provided including Sagnac and relativistic Doppler effects. The frequency shift of photons is also given. We further discuss planetary effets on photon flight time.

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ORBIT AND ENERGY OF A PHOTON IN THE GRAVITATIONAL FIELD OF THE SUN

Let us first assume that only the Sun is present and that it is further spherical and non-rotating. The three spacecraft and photons are thus moving in a static spacetime, the metric of which, at 1st Post-Newtonian level, is

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = (\eta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta$$

$$\sim -\left[1 - 2GM/(rc^2)\right] c^2dt^2 + \left[1 + 2\gamma GM/(rc^2)\right] \delta_{ij}dx^i dx^j \qquad (1)$$

where γ is a Post-Newtonian parameter and M the mass of the Sun.

Let us now consider a photon going from spacecraft A to spacecraft B. The aim is to compute the energy of this photon measured by an observer with velocity v, up to order 3/2 in the small dimensionless parameter $\varepsilon = GM/(rc^2) \sim v^2/c^2 \sim 10^{-8}$. At this order, the latter metric ensures a correct description of the solar gravitational field. Both positions and velocities of A and B are assumed known at initial coordinate time ($t = 0$ = emission time). See Figure 1.

The knowledge of the energy of the photon up to order 3/2 requires knowledge of its orbit up to order 1. It is given by

$$x^i{}_{ph}(t, n^j) = x^i_0 + n^i ct - (\gamma + 1)\frac{GM}{c^2}\chi^i(t; n^j) \qquad (2)$$

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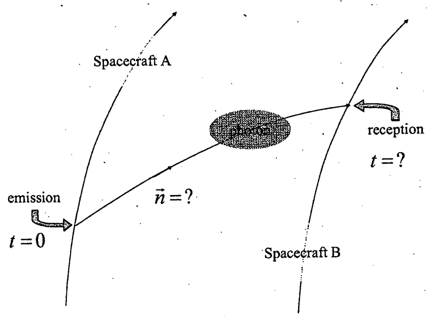

FIGURE 1. Schema of a laser link between spacecraft A and B.

with

$$\chi^i(t,n^j) \equiv P^i[r-r_0] + n^i \ln \frac{\mathbf{n}.\mathbf{x}_0 + ct + r}{\mathbf{n}.\mathbf{x}_0 + r_0}$$

and

$$P^i \equiv \frac{x_0^i - n^i \mathbf{n}.\mathbf{x}_0}{r_0^2 - (\mathbf{n}.\mathbf{x}_0)^2}$$

which satisfies $\mathbf{P}.\mathbf{n} = 0$. The three numbers n^j are integration constants, characterizing the propagation direction. x_0^i is the position of the photon at $t = 0$.

An observer moving at velocity $v^i \equiv dx^i/dt$ measures the energy

$$E \propto 1 - \frac{\mathbf{n}.\mathbf{v}}{c} + \frac{GM}{rc^2} + \frac{\mathbf{v}^2}{2c^2} - \left[(2+\gamma)\frac{GM}{rc^2} + \frac{\mathbf{v}^2}{2c^2}\right]\frac{\mathbf{n}.\mathbf{v}}{c} + (\gamma+1)\frac{GM}{c^2}\frac{\mathbf{P}.\mathbf{v}}{c}\frac{\mathbf{n}.\mathbf{x}}{r}. \tag{3}$$

This requires to compute n^i up to order 1 and v^i up to order 3/2 (the relativistic equation of motion is equivalent to the classical one at this order, since the metric is static).

The detection time and the constants n^i are obtained when the photon and spacecraft positions are identified at the reception time t, using the normalisation equation $\mathbf{n}.\mathbf{n} = 1$. This detection equation reads

$$x_{A,0}^i + n^i ct - (\gamma+1)\frac{GM}{c^2}\chi^i(t,n^j) = x_{B,0}^i + v_{B,0}^i t - \frac{GM x_{B,0}^i}{r_{B,0}^3}\frac{t^2}{2} \tag{4}$$

and the receiver velocity equation up to order 3/2 reads

$$v_B^i(t) = v_{B,0}^i - GM\frac{x_{B,0}^i}{r_{B,0}^3}t - GM\left[\frac{v_{B,0}^i}{r_{B,0}^3} - 3\frac{x_{B,0}^i \mathbf{x}_{B,0}.\mathbf{v}_{B,0}}{r_{B,0}^5}\right]\frac{t^2}{2}, \tag{5}$$

showing that knowledge of the velocity up to order 3/2 requires knowledge of the detection time up to order 1/2. However, the Time Delay Interferometry data pre-processing (TDI, see [1]), crucial in order to reach the gravitational wave detection level, requires to compute the photon flight time up to order 1.

FLIGHT TIME SOLUTION IN THE GRAVITATIONAL FIELD OF THE SUN

In reference [5] details of the computation, up to order 1, of the flight time in the gravitational field of the Sun are provided. We quote here the results. The photon travel time is

$$t = \overset{(0)}{t} + \frac{\overset{(0)}{\mathbf{n}} .\mathbf{v}_{B,0}}{c} \overset{(0)}{t} + \frac{1}{2}\left[\frac{\mathbf{v}_{B,0}^2}{c^2} + \left(\frac{\overset{(0)}{\mathbf{n}} .\mathbf{v}_{B,0}}{c}\right)^2\right] \overset{(0)}{t}$$

$$+ \overset{(0)}{n}{}^i \left[(1+\gamma)\frac{GM}{c^3}\chi^i(\overset{(0)}{t}, \overset{(0)}{n}{}^j) - \frac{GMx^i_{B,0}}{2r^3_{B,0}c} \overset{(0)}{t}{}^2\right] \tag{6}$$

and its so-called direction is provided by the following expression

$$n^i = \overset{(0)}{n}{}^i + \frac{v^i_{B,0}}{c} - \overset{(0)}{n}{}^i \frac{\overset{(0)}{\mathbf{n}} .\mathbf{v}_{B,0}}{c} - \frac{1}{2c^2}\overset{(0)}{n}{}^i\left[\mathbf{v}^2_{B,0} - (\overset{(0)}{\mathbf{n}} .\mathbf{v}_{B,0})^2\right] - \frac{GM\overset{(0)}{t}}{2r^3_{B,0}c}\left[x^i_{B,0} - \overset{(0)}{n}{}^i \overset{(0)}{\mathbf{n}} .\mathbf{x}_{B,0}\right]$$

$$+(\gamma+1)\frac{GM}{c^3\overset{(0)}{t}}\left[\chi^i - \overset{(0)}{n}{}^i \overset{(0)}{\mathbf{n}} .\chi\right]\left(\overset{(0)}{t}, \overset{(0)}{n}{}^k\right) \tag{7}$$

where the two 0-th order terms are

$$c\overset{(0)}{t} = \pm\sqrt{(x^i_{B,0} - x^i_{A,0})(x^i_{B,0} - x^i_{A,0})} \tag{8}$$

and

$$\overset{(0)}{n}{}^i = \frac{x^i_{B,0} - x^i_{A,0}}{c\overset{(0)}{t}} \tag{9}$$

The + (/-) sign corresponds to a signal propagating from A to B (B to A).

Graphs showing the contribution of orders 0, 1/2 and 1 to the photon travel time from spacecraft A to B are given in reference [5], together with the Sagnac and aberration contributions (order 1/2).

FREQUENCY SHIFT SOLUTION IN THE GRAVITATIONAL FIELD OF THE SUN

In reference [5] details of the computation, up to order 3/2, of the frequency shift solution in the gravitational field of the Sun are provided. We quote here the results, assuming that the photon emitted by A is received by B.

The frequency shift can be defined as

$$z = \frac{E_{rec}}{E_{em}} - 1 = \overset{(1/2)}{z} + \overset{(1)}{z} + \overset{(3/2)}{z} + \ldots \tag{10}$$

A naive estimate would lead to

$$\overset{(k)}{z} \sim \delta\left(\varepsilon^k\right) \sim \varepsilon^k \frac{L}{2r}$$

where L is the LISA arm length; hence $\overset{(1/2)}{z} \sim 2.10^{-6}$, $\overset{(1)}{z} \sim 2.10^{-10}$ and $\overset{(3/2)}{z} \sim 2.10^{-14}$. But in fact, since $\mathbf{n}.\mathbf{v}_{AB,0} = \overset{(0)}{n}{}^i \, (v^i_{B,0} - v^i_{A,0}) \sim 0$ due to LISA's configuration, the 1/2th order contribution ends up to

$$\overset{(1/2)}{z} = -\mathbf{n}.\frac{\mathbf{v}_{AB,0}}{c} \sim 7.10^{-8}. \tag{11}$$

The first order contribution

$$\overset{(1)}{z} = \left(\frac{\overset{(0)}{\mathbf{n}}.\mathbf{v}_{AB,0}}{c}\right)^2 - \frac{1}{2}\left(\frac{\mathbf{v}_{AB,0}}{c}\right)^2$$

$$+\frac{GM}{c}\overset{(0)}{t}\frac{\overset{(0)}{\mathbf{n}}.\mathbf{x}_{B,0}}{r^3_{B,0}} + \frac{GM}{c^2}\left(\frac{1}{r_{B,0}} - \frac{1}{r_{A,0}}\right) \tag{12}$$

is also significantly reduced. Indeed, the first term (kinematic contribution) amounts to only $\sim 2.10^{-15}$, the second term (kinematic contribution) is $\sim 2.10^{-12}$ while the third term (acceleration of the receiving spacecraft during the flight of the photon) or the fourth term (Einstein gravitational shift) are $\sim 2.10^{-10}$ separately. But the second, third and fourth terms nearly completely compensate howing to LISA's configuration, so that finally $\overset{(1)}{z} \sim 2.10^{-13}$.

Graphs showing the contribution of orders 1/2 and 1 to the frequency shift of a photon travelling from spacecraft A to B are given in reference [5].

PLANETARY EFFECTS ON PHOTON FLIGHT TIME

We now consider the presence of planets and a non-sherical rotating Sun. The perturbation from a flat Minkowskian metric is given by

$$h_{\alpha\beta} = \overset{(1)}{h}_{\alpha\beta} + \overset{(3/2)}{h}_{\alpha\beta} + \overset{(2)}{h}_{\alpha\beta} + \ldots$$

The first order contribution to $h_{\alpha\beta}$ can be separated into the Sun's contribution and that from planets. At this order, the Sun contributes through its position ($\sim 10^{-8}$) and its change in position during the flight of photon ($\sim 2.10^{-16}$) but its flattening is negligible (its quadrupole moment is of the order of $\sim 10^{-7}$). The planets, taken as mass points, contribute through their positions ($\sim 2.10^{-12}$) and their change in position during the flight of photon ($\sim 10^{-18}$).

In the 3/2-th contribution to $h_{\alpha\beta}$, the solar rotation amounts to less than $\sim 10^{-13}$ and its orbital velocity to $\sim 10^{-15}$. The planets, taken as mass points, contribute by their orbital velocity ($\sim 2.10^{-16}$ for Jupiter or $\sim 10^{-17}$ for Venus).
The second order contribution to $h_{\alpha\beta}$ is about the square of the first order contribution due to a static spherical Sun.
Since trajectory of a photon is $\sim 5.10^6$ km, we see that planetary presence, solar asphericities or rotation perturbations are negligible for photon trajectories. The meaningfull contribution ($\sim 10^{-8}.10^6 km \sim 50m \sim 2.10^{-7}s$) being that of a spherical Sun which we discussed in the previous sections. However, planets should be taken into account in spacecraft ephemerides which are required to compute flight times ($x_0 = x_{A,0}$, $x_{B,0}$, $v_{B,0}$ and $v_{A,0}$ are needed in the above analytical expressions). Such spacecraft ephemerides are under study.

LISACODE, A LISA SIMULATOR

The feasability and the evaluation of the LISA mission performances can only be studied via computer simulations of the different processes involved. A LISA simulation code (LISACode) is being developped within the LISA-France collaboration. It includes the analytical expressions for the photon flight time and frequency shifts discussed in the present paper. A presentation of this code can be found on the following website: http://www.apc.univ-paris7.fr/LISA-France/analyse.phtml. In [4], LISACode is compared with two other simulators elaborated in the US: Synthetic LISA [2] and LISA simulator [3].

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