To the readers…

In order to improve these notes, please report any errors that you may encounter (there might be some!) to

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I shall appreciate your comments…

Centre de formation IGN, Forcalquier, 30/08/04 – 04/09/04

Deuxième école d'été du GRGS:

Géodésie spatiale, Physique de la Mesure et Physique Fondamentale

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Conventions:

Usual partial derivative with $\frac{\partial \bullet}{\partial x^{\mu}} = \bullet_{,\mu}$

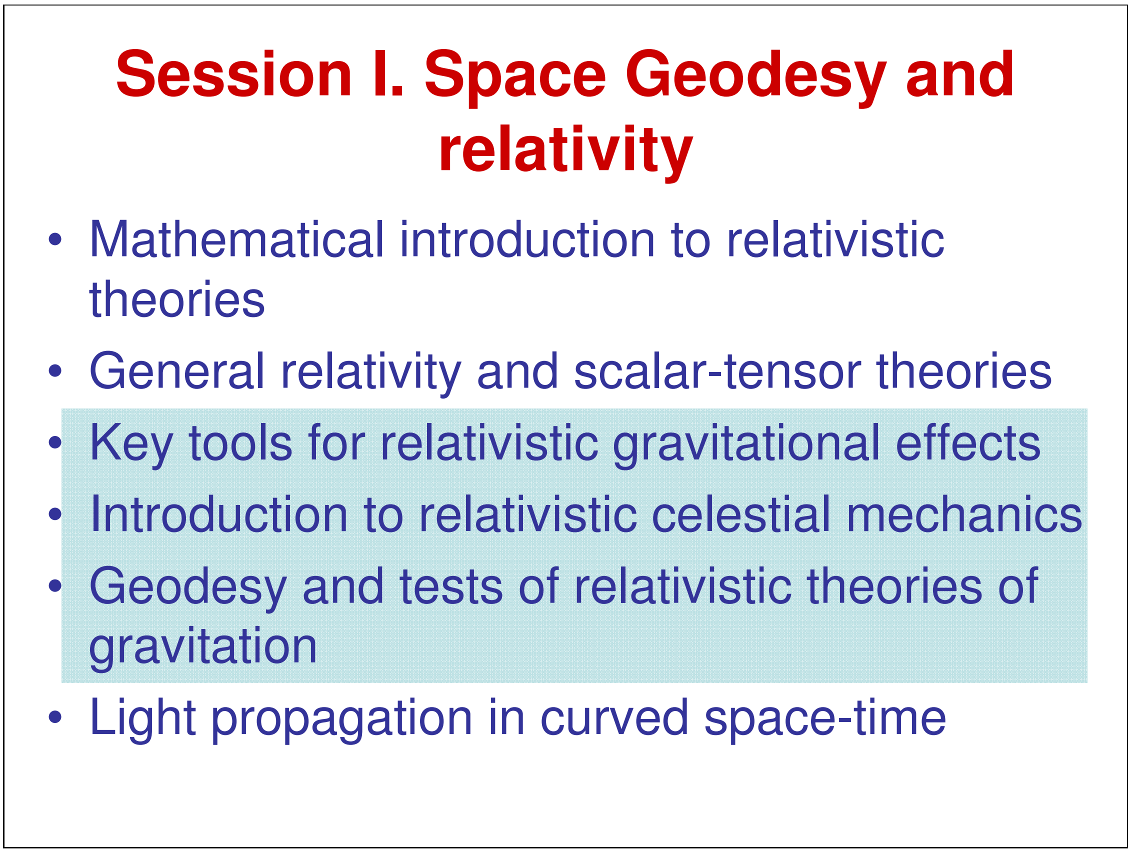
Covariant derivative with $\bullet_{|\mu}$

Latin indexes for 1,2,3 (spatial coordinates)

Greek indexes for 0,1,2,3 (space-time coordinates)

Einstein's convention for indexes summation:

$$\bullet^{\mu} \bullet_{\mu} = \sum_{\mu} \bullet^{\mu} \bullet_{\mu}$$



Session I. Space Geodesy and relativity

- Mathematical introduction to relativistic theories
- General relativity and scalar-tensor theories
- Key tools for relativistic gravitational effects
- Introduction to relativistic celestial mechanics
- Geodesy and tests of relativistic theories of gravitation
- Light propagation in curved space-time

A/ Key tools for relativistic gravitational effects

- Parametrized Post-Newtonian (PPN) formalism
- Constancy of the Newtonian gravitational constant (G)
- IAU resolutions on reference systems

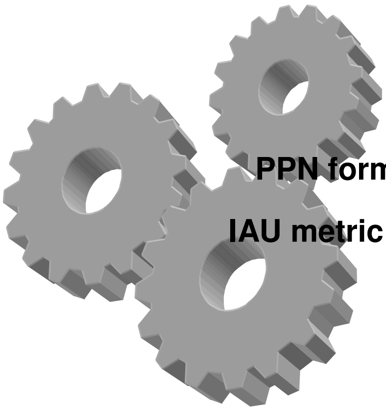
B/ Introduction to relativistic celestial mechanics

- Two body system in weak field approximation
- Main relativistic effects in celestial mechanics and tests of relativistic theories of gravitation

C/ Geodesy and tests of relativistic theories of gravitation

- Geodetic satellite equations of motion
- Main relativistic effects in geodesy
- Satellite tests of relativistic theories of gravitation

A/ Key tools for relativistic gravitational effects



RELATIVISTIC THEORIES OF GRAVITATION

PPN formalism

IAU metric conventions

A0. Key features in relativistic gravitation

- **Relevance of the metric:**

Given be coordinates $x^\mu = (ct, x, y, z)$, or alternatively $X^\mu = (cT, X, Y, Z)$,

$$c^2\, d\tau^2 = ds^2 = g_{\mu\nu}(x^\mu) \cdot dx^\mu \cdot dx^\nu = G_{\mu\nu}(X^\mu) \cdot dX^\mu \cdot dX^\nu$$

proper time

metric

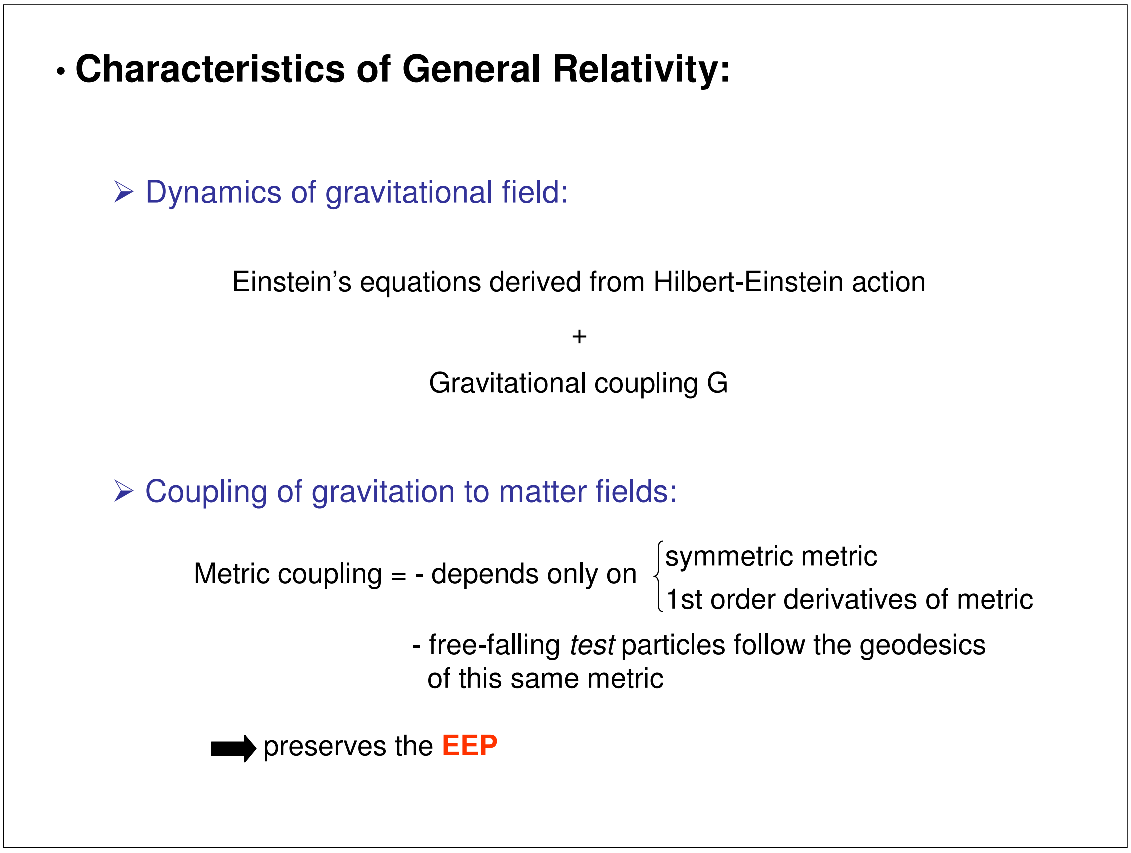
Invariant line-element

coordinate time and space

➡ Relation between τ and t: relativistic time dilation integral

➡ Relation between t and x, y, z : (Geodesic) equations of motion
(if only gravitational forces)

➡ Relation between x^μ and X^μ: coordinate space-time transformations



• Characteristics of General Relativity:

- Dynamics of gravitational field:

Einstein’s equations derived from Hilbert-Einstein action

+

Gravitational coupling G

- Coupling of gravitation to matter fields:

Metric coupling = - depends only on $\begin{cases}\text{symmetric metric}\\ \text{1st order derivatives of metric}\end{cases}$

- free-falling *test* particles follow the geodesics of this same metric

➡ preserves the **EEP**

G = physically measured Newtonian constant far from the Solar System

Test particle = point-like particle

Free falling particle = particle under the influence of the sole gravitational force

Einstein Equivalence Principle (EEP) = postulates

1. $m_{grav} = m_I$ for test (point-like) particles, independently of their mass and composition.

➡ **Weak Equivalence Principle (WEP)** = universality of Free Fall for particles with $E_{grav} \approx 0$

2. equivalence, locally, of any free-falling reference frame with a reference frame at rest in empty space, from the point of view of all *non-gravitational* experiments for particles with $E_{grav} \approx 0$
In other words:

2a. **Local Lorentz Invariance Principle (LLIP)** = in a freely-falling reference frame, locally, the result of any *non-gravitational* experiment is independent of the velocity of the reference frame.

➡ isotropy of c_0 (∄ preferred referential rest frames).

2b. **Local Positional Invariance Principle (LPIP)** = in a freely-falling reference frame, locally, the result of any *non-gravitational* experiment is independent of the space-time point where this experiment is made.

➡ - constancy of Standard Model (SM) "constants" (they depend only on coupling constants and mass scales in SM).

- universal gravitational redshift. When two identical clocks are located at different positions in a static external Newtonian potential (U), they show, independently of their nature and constitution, a difference in clock rate when compared thanks to electromagnetic signals $\propto \Delta U / c^2$.

m_{grav} = ***gravitational* mass** (the mass that determines the gravitational potential exhibited by a body or mass that determines the force on a body in a gravitational field.)

m_I = ***inertial* mass**, the mass of the body to which the resulting acceleration (gravitational or non gravitational) is applied, in Newton's 1st law of motion.

E_{grav} = **- *gravitational* binding energy**

c_0 = speed of light

c = structure constant appearing in the metric and gravitational equations. In General Relativity, c is the speed of light, but there are alternative theories for which it is not the case.

- **General Relativity further verifies**

Strong Equivalence Principle (SEP) = postulates

1. $m_{grav} = m_I$ for test particles and **bodies**, independently of their mass and composition.

➡ **Gravitational WEP (GWEP)** = WEP extended to bodies with $E_{grav} \not\approx 0$

2. equivalence, locally, of any free-falling reference frame with a reference frame at rest in empty space, from the point of view of all *non-gravitational* and ***gravitational*** experiments for particles with $E_{grav} \approx 0$ or $E_{grav} \not\approx 0$. (**GLLIP**, **GLPIP**)

SEP $\supset$ EEP

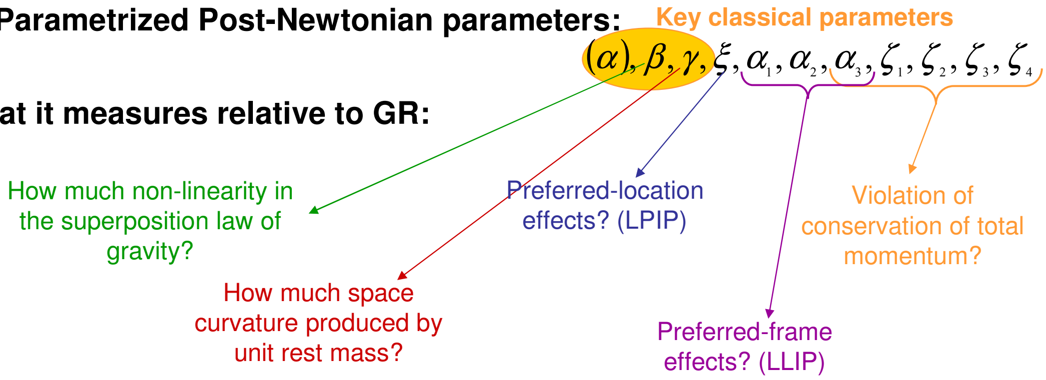
- **Alternative theories to General Relativity violate SEP…**

<u>ex:</u> TS theories, **~~GLPIP~~** ➡ $\dot{G} \neq 0$

locally

A1. PPN formalism

- **Use:** characterize different theories of gravitation for tests via their metric
- **Authors:** Eddington, Robertson and Schiff; Nordtvedt; Will and Nordvedt
- **10 Parametrized Post-Newtonian parameters:**
- **What it measures relative to GR:**



Key classical parameters

$(\alpha), \beta, \gamma, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4$

How much non-linearity in the superposition law of gravity?

How much space curvature produced by unit rest mass?

Preferred-location effects? (LPIP)

Preferred-frame effects? (LLIP)

Violation of conservation of total momentum?

➡ in GR: $\alpha=\beta=\gamma=1$ and others $=0$

➡ in semi-conservative theories: $\alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4=0$

➡ in fully-conservative theories: $\alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4=0$

In TS theories: $\alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4\big|_{TS}=0$ (fully conservative theory)

$\xi_{TS}=0$ no preferred location effect

$$\alpha_{TS}=1$$

$$\gamma_{TS}=\frac{1+\omega_0}{2+\omega_0}$$

with

$$\beta_{TS}=1+\Lambda_0$$

$$\omega_0 \equiv \omega(\phi_0)$$

$$\Lambda_0 \equiv \frac{d\omega/d\phi}{(3+2\omega)^2(4+2\omega)}\bigg|_{\phi_0}$$

ϕ_0 = asymptotic (when corresponding metric is Minkowskian) value of the scalar field, may vary on a Hubble time scale as the universe evolves.
Hence, G may vary too.

$$G_{TS\ today}=\phi_0^{-1}\frac{4+2\omega_0}{3+2\omega_0}$$

Remark: Significance of β, the coefficient of $U/c^2=GM/rc^2$, as amount of non-linearity (GM/rc^2) introduced in the g_{00}-metric is heuristic because the presence of β-term is gauge dependant. See for example the transformation between Schwarzschild and isotropic coordinates for the Schwarzschild metric.
The significance of γ is unambiguous because the Riemannian curvature tensor $R_{ijkl} \propto \gamma$ in any gauge.

- **Metric:** $$g_{00} = -1 + 2\alpha\frac{U}{c^2} - 2\beta\left(\frac{U}{c^2}\right)^2 - 2\xi\frac{\Phi_W}{c^2}$$
$$+ (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\frac{\Phi_1}{c^2} + 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\frac{\Phi_2}{c^2}$$
$$+ 2(1 + \zeta_3)\frac{\Phi_3}{c^2} + 2(3\gamma + 3\zeta_4 - 2\xi)\frac{\Phi_4}{c^2}$$
$$-(\zeta_1 - 2\xi)\frac{A}{c^2} - (\alpha_1 - \alpha_2 - \alpha_3)\left(\frac{w}{c}\right)^2\frac{U}{c^2} - \alpha_2\frac{w^i}{c}\frac{w^j}{c}\frac{U_{ij}}{c^2} + (2\alpha_3 - \alpha_1)\frac{w^i}{c}\frac{V_i}{c^2}$$
$$+ \Theta(\varepsilon^3)$$

$$g_{0i} = -\frac{1}{2}(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)\frac{V_i}{c^2} - \frac{1}{2}(1 + \alpha_2 - \zeta_1 + 2\xi)\frac{W_i}{c^2}$$
$$-\frac{1}{2}(\alpha_1 - 2\alpha_2)\frac{w^i}{c}\frac{U}{c^2} - \alpha_2\frac{w^j}{c}\frac{U_{ij}}{c^2}$$
$$+ \Theta(\varepsilon^{5/2})$$

Key classical parameters

$$g_{ij} = \left(1 + 2\gamma\frac{U}{c^2}\right)\delta_{ij} + \Theta(\varepsilon^2)$$

- **Metric potentials:** $U, U_{ij}, \Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, A, V_i, \mathrm{W}_i$ in Box 2 p30 [Will 2001]

Newtonian potential

$$\boxed{U \equiv G\int \frac{\rho'}{|\vec{x}' - \vec{x}|} d^3x'}$$

- **Matter variables:** ρ, p = density, pressure of rest-mass in local freely falling frame
  momentarily moving with matter
  v^i = coordinate matter velocity
  w^i = coordinate velocity of PPN coordinate system
  1/1 mean rest-frame of the Universe
  Π = internal energy per unit rest-mass.
  non-rest-mass, non gravitational energy

- **Stress energy tensor:** $T^{00} = \rho c^2 \left(1 + \frac{\Pi}{c^2} + \frac{v^2}{c^2} + 2\frac{U}{c^2}\right)$

(perfect fluid)

$$T^{0i} = \rho c\, v^i \left(1 + \frac{\Pi}{c^2} + \frac{v^2}{c^2} + 2\frac{U}{c^2} + \frac{p}{\rho c^2}\right)$$

$$T^{ij} = \rho\, v^i\, v^j \left(1 + \frac{\Pi}{c^2} + \frac{v^2}{c^2} + 2\frac{U}{c^2} + \frac{p}{\rho c^2}\right) + p\,\delta^{ij}\left(1 - 2\gamma\frac{U}{c^2}\right)$$

Energy:

In $\rho c^2 \left(1 + \frac{\Pi}{c^2} + \frac{v^2}{c^2} + 2\frac{U}{c^2}\right)$

the different terms are rest mass energy density, thermal energy, kinetic energy, gravitational energy and pressure energy.

Metric potentials (*c*=*G*=*1*):

$$U \equiv \int \frac{\rho'}{|\vec{x} - \vec{x}'|} d^3x' \qquad U_{ij} \equiv \int \frac{\rho' (\mathrm{\vec{x}} - \mathrm{\vec{x}}')_i (\mathrm{\vec{x}} - \mathrm{\vec{x}}')_j}{|\vec{x} - \vec{x}'|^3} d^3x'$$

$$\Phi_W \equiv \int \frac{\rho' \rho'' (\mathrm{\vec{x}} - \mathrm{\vec{x}}')}{|\vec{x} - \vec{x}'|^3} \cdot \left(\frac{(\mathrm{\vec{x}}' - \mathrm{\vec{x}}'')}{|\vec{x} - \vec{x}''|} - \frac{(\mathrm{\vec{x}} - \mathrm{\vec{x}}'')}{|\vec{x}' - \vec{x}''|}\right) d^3x'\, d^3x''$$

$$A \equiv \int \frac{\rho' \, [\vec{v}' \cdot (\mathrm{\vec{x}} - \mathrm{\vec{x}}')]^2}{|\vec{x} - \vec{x}'|^3} d^3x' \qquad \chi \equiv \int \rho' |\vec{x} - \vec{x}'|\, d^3x'$$

$$\Phi_1 \equiv \int \frac{\rho' \, \vec{v}'^2}{|\vec{x} - \vec{x}'|} d^3x' \qquad \Phi_2 \equiv \int \frac{\rho' \, \mathrm{U}'}{|\vec{x} - \vec{x}'|} d^3x'$$

$$\Phi_3 \equiv \int \frac{\rho' \, \Pi'}{|\vec{x} - \vec{x}'|} d^3x' \qquad \Phi_4 \equiv \int \frac{p'}{|\vec{x} - \vec{x}'|} d^3x'$$

$$V_i \equiv \int \frac{\rho' \, \vec{v}'_i}{|\vec{x} - \vec{x}'|} d^3x' \qquad W_i \equiv \int \frac{\rho' [\vec{v}' \cdot (\vec{x} - \vec{x}')] (\vec{x} - \vec{x}')_i}{|\vec{x} - \vec{x}'|^3} d^3x'$$

• PPN metric Gauge:

a) PPN coordinates:

Hyp:
- universe homegeneous
  isotropic
- isolated PPN system: far outer regions - in free fall 1/1 cosmological model
  - at rest 1/1 universe rest-frame

PPN coord: local quasi cartesian coordinates x^{μ} for which

- time variation of comological part of metric negligible 1/1 solar system exp.
- $ds^{2} = (\eta_{\mu\nu} + h_{\mu\nu}) \cdot dx^{\mu} \cdot dx^{\nu}$ with $h_{\mu\nu}$ = PPN-part >> cosmological effect (RW scale factor)
- x^{μ} asymptotically Minkowskian in outer regions

Universe rest-frame = frame in which universe is isotropic and homogeneous

RW = Robertson-Walker

b) PPN Potentials: built from combinations of matter variables so that

i. $h_{\mu\nu} \ni$ only Newtonian or Post-Newtonian terms

ii. $h_{\mu\nu} \xrightarrow[|\vec{x}-\vec{x}'|\to\infty]{} 0$ to guarantee Minkowskian asymptotic limit

space-time pt pt inside matter

iii. $g_{\mu\nu}$ is dimensionless

iv. arbitrary space-time origin ➡ $h_{\mu\nu}$ - built on functionals of $|\vec{x}-\vec{x}'|$
- $\ni$ explicit time dependency in matter variables and matching PPN parameters

v. $h_{\mu\nu}$ transforms under space rotations as:
00 ⟶ scalar
0i ⟶ vector
ij ⟶ tensor

vi. $g_{\mu\nu}$ - $\not\ni$ $\vec{\nabla}\rho, \vec{\nabla}p, \vec{\nabla}Energy, \vec{\nabla}m$
- $\ni$ simple functionals
(arbitrary requirement)

c) PPN gauge:

Infinitesimal coordinate/gauge transformation:

$$x^{\mu} \curvearrowright \tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}(x^{\nu}) \quad \Rightarrow \quad g_{\mu\nu} \curvearrowright \tilde{g}_{\mu\nu} = g_{\mu\nu}(\tilde{x}^{\mu}) - \xi_{\mu|\nu} - \xi_{\nu|\mu}$$

Gauge choice: SPN … for convenience, no physical meaning

- $\tilde{g}_{\mu\nu}$ should still verify: Post-Newtonian, quasi-Cartesian coordinates, universe rest-frame

$$\Rightarrow \quad \xi_{\mu} = \lambda_{\mu} \frac{\partial \chi}{\partial x^{\mu}} \qquad \text{with} \begin{cases} \lambda_0 = \lambda_1 \\ \lambda_{\mu=1,2,3} = \lambda_2 \end{cases} \text{still arbitrary}$$

$$\Rightarrow \quad \begin{aligned} \tilde{g}_{00} &= g_{00} + \dots \\ \tilde{g}_{0i} &= g_{0i} + \dots \\ \tilde{g}_{ij} &= g_{ij} + \dots \end{aligned}$$

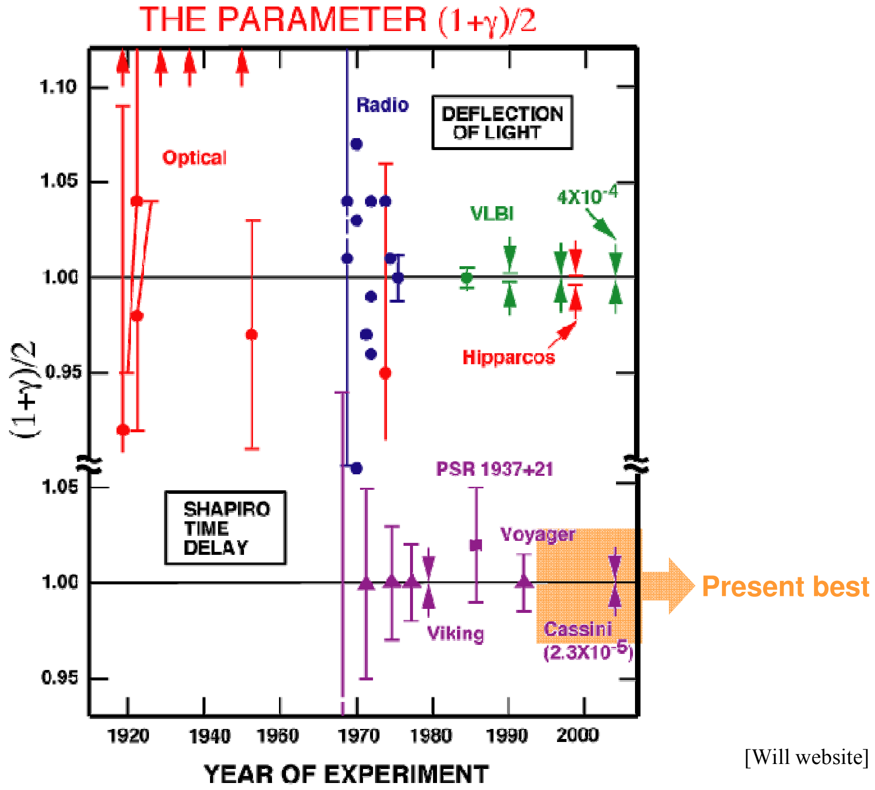
functions of λ_1, λ_2, PPN potentials for x^{μ}

- Choose λ_1, λ_2 so that $\begin{cases} \tilde{g}_{00} \not\ni \int \frac{\rho'}{|\vec{x} - \vec{x}'|} (\vec{x} - \vec{x}') \cdot \frac{d\vec{v}'}{dt} d^3x' \\ \tilde{g}_{ij} < \text{diagonal, isotropic} \end{cases}$

$\Rightarrow$ for a given theory of gravitation, PPN coefficients are unique

SPN = Standard Post Newtonian gauge

- **Tests of gravitation:**



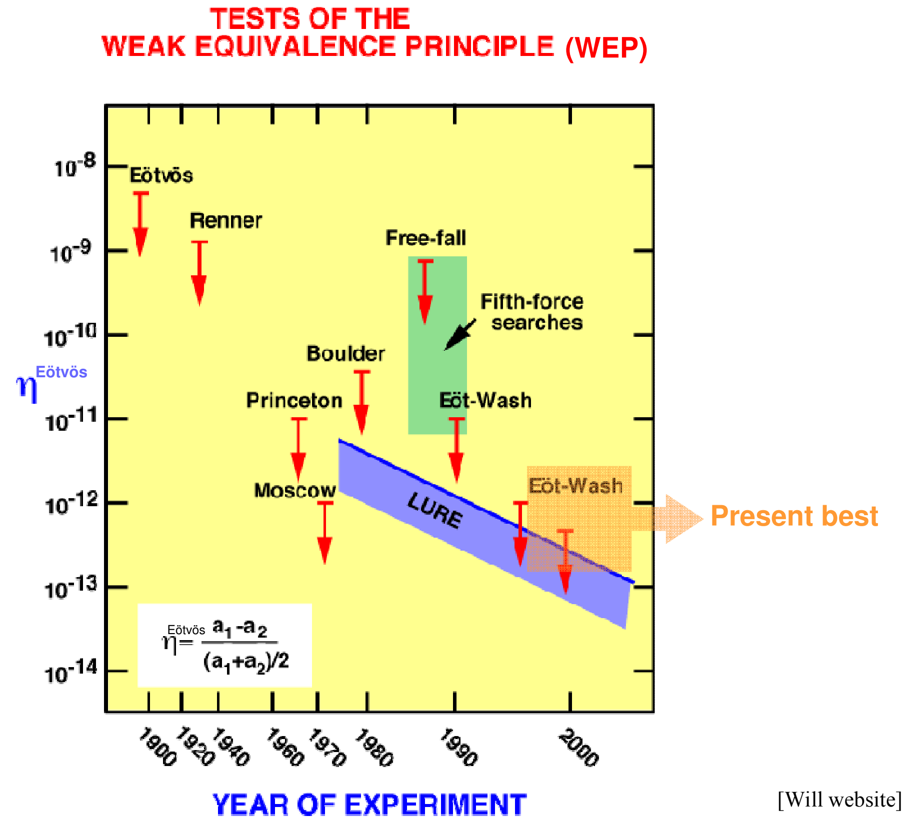
[Will website]

In red: measurement of deflection of optical starlight. Arrows in upper left corner are for early eclipse measurements (anomalously large values).

In blue: radio interferometry using quasars.

In green: **VLBI** (Very Long Baseline Interferometry) measurements.

In purple: radar measurement of (Shapiro) **time delay**.



[Will website]

Many experiments in 1985-90 searched for a '**fifth force**' in nature (Yukawa type of potential, finite range) inducing composition-dependent difference in acceleration. Those were reinterpreted as WEP experiment in the limit of an infinite range.

Universality of Free Fall (WEP) for tests particles has been verified by pendulum, torsion balances (Eötvös, Renner, Princeton, Moscow, …); flotation on water (Boulder); comparing accelerations of various materials towards local topography of the Earth, movable laboratory masses, the Sun, the Galaxy (for extended bodies, it is in fact GWEP which contains WEP).
In the future, space missions like MICROSCOPE (2008), STEP (Satellite Test of the Equivalence Principle) (phase A) should test WEP to respectively 10^{-15} and 10^{-13} - 10^{-18} ; (space) clocks working with different types of atoms could also be used [GRGS school 2004] .

Eötvös-type experiments = comparison of the acceleration from rest of two laboratory-size objects of different compositions in an external gravitational field. Test WEP.

$$\frac{m_P}{m_I} = 1 - \sum_{\text{interaction grav, nongrav}} \eta^{\text{interaction}} \cdot \frac{E_{\text{interaction}}}{m_I c^2} \quad \text{and} \quad m_I \cdot a = m_P \cdot g$$

with $E_{\text{interaction}} =$ internal energy due to a given interaction

…verify if same a for different test bodies 1, 2:

$$\eta^{\text{Eötvös}} = \frac{2\,|a_1 - a_2|}{|a_1 + a_2|}$$

LURE (Lunar Laser Ranging experiment) = tests of the equality of acceleration for the Earth and Moon toward the Sun (Nordtvedt effect).
Is not a pure WEP test, because those are extended bodies.

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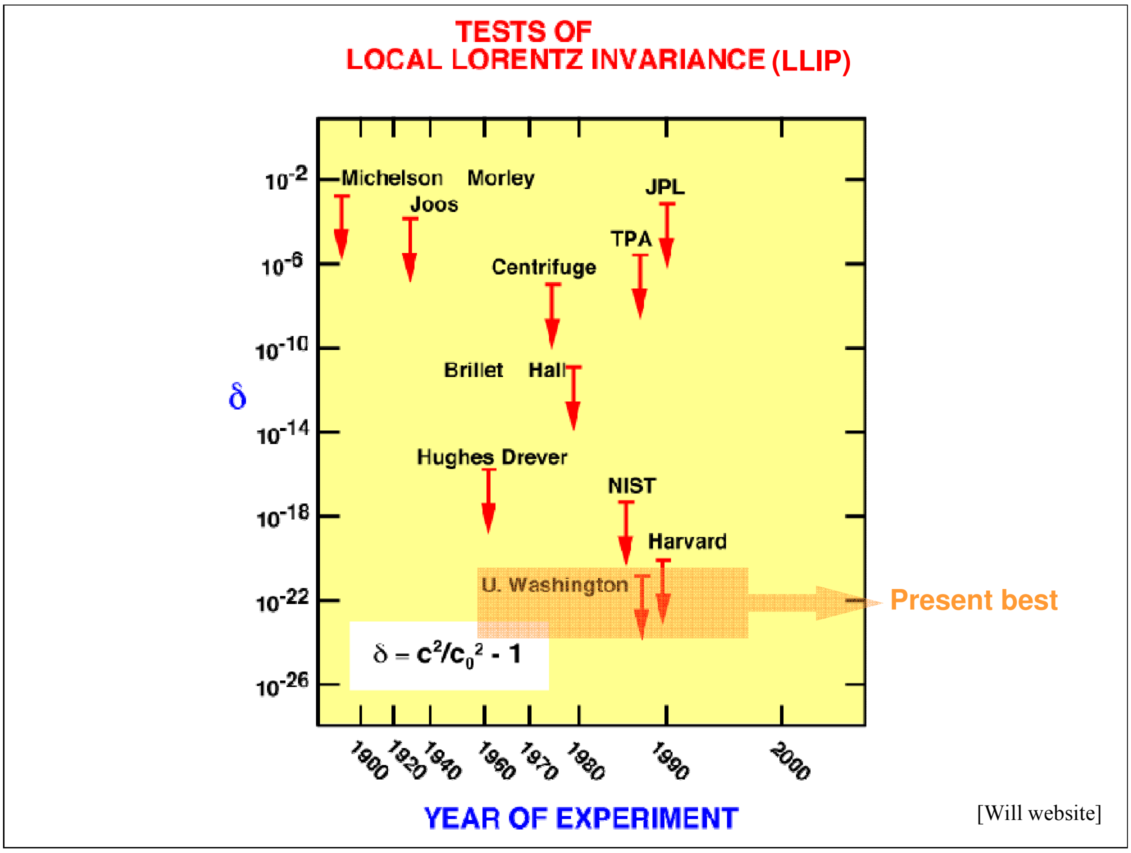


[Will website]

Experiments testing the equality between the speed of electromagnetic waves and the limiting speed of test bodies.

The most precise experiments bound anisotropies in atomic energy level as a violation of LLIP would have direct incidence on the energy levels in atomic nuclei.

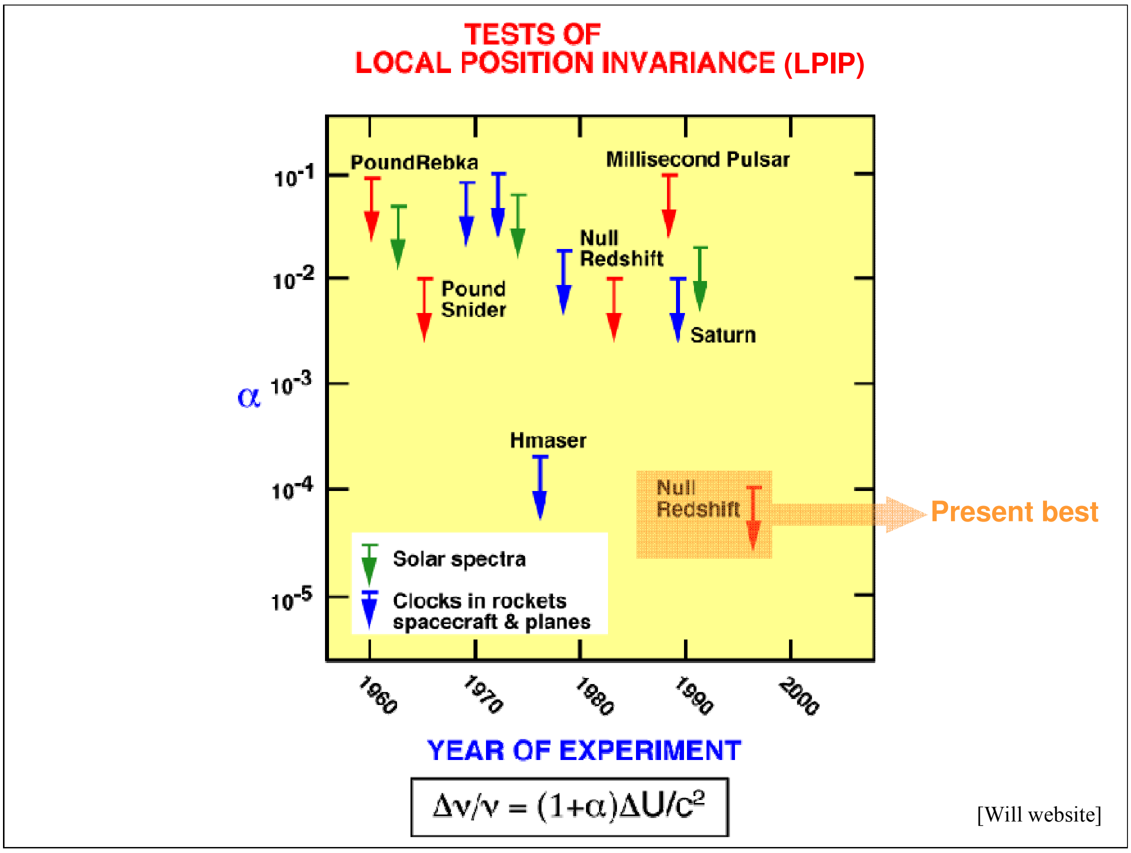
c_0 = speed of light

c = structure constant appearing in the metric and gravitational equations. In General Relativity, c is the speed of light, but there are alternative theories for which it is not the case.

Future ACES (Atomic Clock Ensemble in Space) mission will test the anisotropy of the speed of light at

$$\frac{\delta c_0}{c_0} < 10^{-10} \qquad \text{(by comparison, GPS tests:} \qquad \frac{\delta c_0}{c_0} < qq\,10^{-9} \text{)}$$

by comparing time with earth and space-bound clocks, as a function of their respective orientation [GRGS school 2004] .



[Will website]

Sharp constraints for **time-LLIP** have been set on the variability of Standard Model constants using very different data sources (spectral lines in galaxies, natural fission reactor in Gabon 2 billion years ago…). Example of the most stringent limit on the fine structure constant:

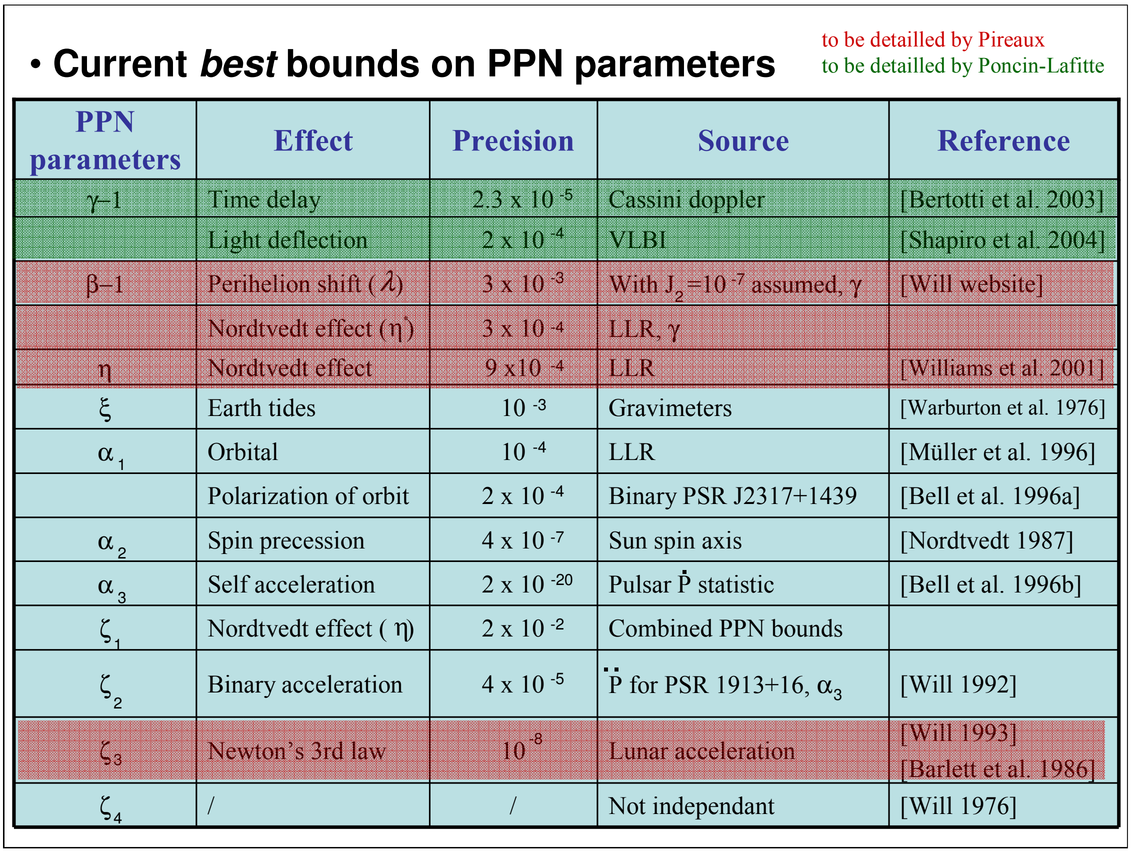
$$\left|\dot{\alpha}_{SM}/\alpha_{SM}\right| < 10^{-15}\ yr^{-1}$$ [Will 2001]

Future ACES (Atomic Clock Ensemble in Space) should look at a drift in the fundamental constants down to 10^{-16}/year [GRGS school 2004] .

As for **space-LLIP**, they test the universality of the gravitational redshift:

Hmaser = 1976 NASA-SAO rocket redshift experiment using a hydrogen-maser clock

Null redshift experiments = experiment which intercompare clocks of different types in the varying gravitational potential.
Future ACES should test the universality of redshift at 10^{-6}. [ACES 2004]



- **Current *best* bounds on PPN parameters**

to be detailled by Pireaux
to be detailled by Poncin-Lafitte

PPN parameters	Effect	Precision	Source	Reference
$\gamma-1$	Time delay	2.3×10^{-5}	Cassini doppler	[Bertotti et al. 2003]
	Light deflection	2×10^{-4}	VLBI	[Shapiro et al. 2004]
$\beta-1$	Perihelion shift (λ)	3×10^{-3}	With $J_2=10^{-7}$ assumed, γ	[Will website]
	Nordtvedt effect (η^*)	3×10^{-4}	LLR, γ	
η	Nordtvedt effect	9×10^{-4}	LLR	[Williams et al. 2001]
ξ	Earth tides	10^{-3}	Gravimeters	[Warburton et al. 1976]
α_1	Orbital	10^{-4}	LLR	[Müller et al. 1996]
	Polarization of orbit	2×10^{-4}	Binary PSR J2317+1439	[Bell et al. 1996a]
α_2	Spin precession	4×10^{-7}	Sun spin axis	[Nordtvedt 1987]
α_3	Self acceleration	2×10^{-20}	Pulsar $\dot{P}$ statistic	[Bell et al. 1996b]
ζ_1	Nordtvedt effect (η)	2×10^{-2}	Combined PPN bounds	
ζ_2	Binary acceleration	4×10^{-5}	$\ddot{P}$ for PSR 1913+16, α_3	[Will 1992]
ζ_3	Newton's 3rd law	10^{-8}	Lunar acceleration	[Will 1993] [Barlett et al. 1986]
ζ_4	/	/	Not independant	[Will 1976]

$\lambda \equiv (2\alpha + 2\gamma - \beta)/3$ Classical PPN combination in perihelion shift

$$\eta^* \equiv 4\beta - \gamma - 3$$

$\eta \equiv 4\beta - \gamma - 3 - 10\xi/3 - \alpha_1 + 2\alpha_2/3 - 2\zeta_1/3 - \zeta_2/3$ Nordtvedt parameter

$\zeta_4 = (3\alpha_3 + 2\zeta_1 - 3\zeta_3)/6$

As in any reasonable theory of gravity, there should be a connection between gravity produced by kinetic energy, internal energy and pressure (to which ζ_4 is related). [Will 1976]

LLR = Lunar Laser Ranging

[Bertotti et al. 2003] obtained $\gamma - 1 = (-2.1 \pm 2.3)\,10^{-5}$ from Doppler (Cassini spacecraft around a solar conjunction period)

[Shapiro et al. 2004] obtained $\gamma = 0.99983 \pm 0.00045$ from 1.7 million ionosphere- corrected group delay measurement from 87 VLBI observatories.

[Lebach et al. 1995] obtained $\gamma = 0.9996 \pm 0.0017$ using quasars 3C273 and 3C279 with dedicated VLBI measurements.

[Eubanks et al. 1999] obtained $\frac{1+\gamma}{2} = 0.99992 \pm 0.00014$ unpublished, cited [Will 1993]

Other less stringent estimates of γ where obtained in the past from VLBI data:
[Seielstad et al. 1970], [Counselman et al. 1974], [Fomalont et al. 1976], [Robertson et al. 1984], [Robertson et al. 1991a], [Robertson et al. 1991b].

- **Application to TS theories with free parameter $\omega(\phi)$:**

a) Values of PPN parameters

$\alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \Big|_{TS} = 0$ ➡ fully conservative theory

$\xi_{TS} = 0$ ➡ no preferred location effect

$\alpha_{TS} = 1$

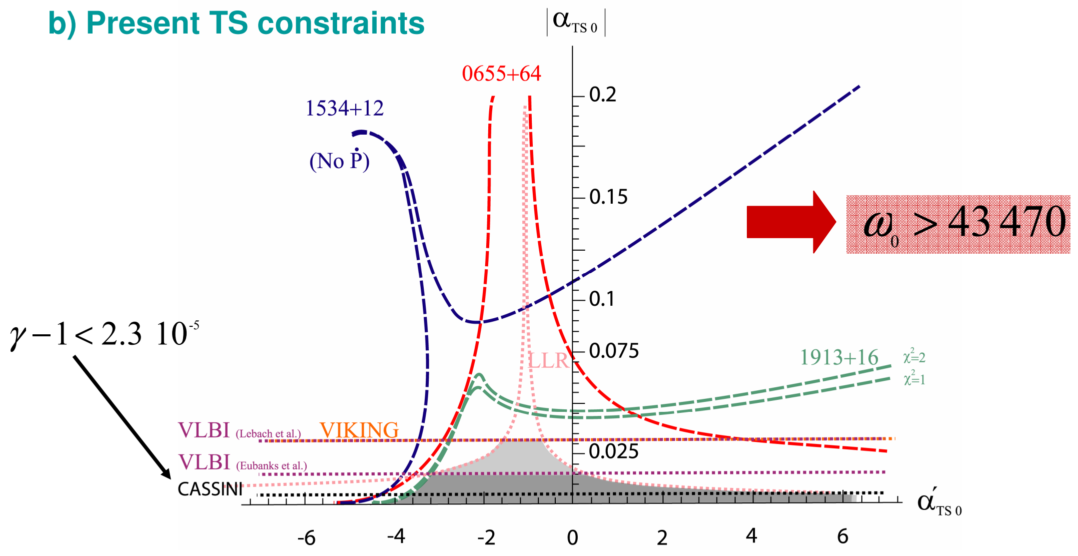
$\gamma_{TS} = \dfrac{1+\omega_0}{2+\omega_0} \leq 1 \equiv \gamma_{GR}$ ➡ less deflection, time-delay than GR

$\beta_{TS} = 1 + \Lambda_0$ with $\begin{cases} \omega_0 \equiv \omega(\phi_0) \\ \Lambda_0 \equiv \dfrac{d\omega / d\phi}{(3+2\omega)^2 (4+2\omega)} \Bigg|_{\phi_0} \\ \phi_0 \equiv \text{asymptotic value of } \phi \end{cases}$

may vary on a H₀ time scale as universe evolves

$G_{TS\,today} = \phi_0^{-1} \dfrac{4+2\omega_0}{3+2\omega_0}$ ➡ ***G* may vary**

b) Present TS constraints



Where $\alpha_{TS} = \dfrac{1}{\sqrt{2\omega(\phi)+3}}$ = the strength of coupling ϕ - matter

$$\alpha'_{TS} \equiv d\alpha_{TS}/d\phi$$

$_0 \equiv$ taken at ϕ_0

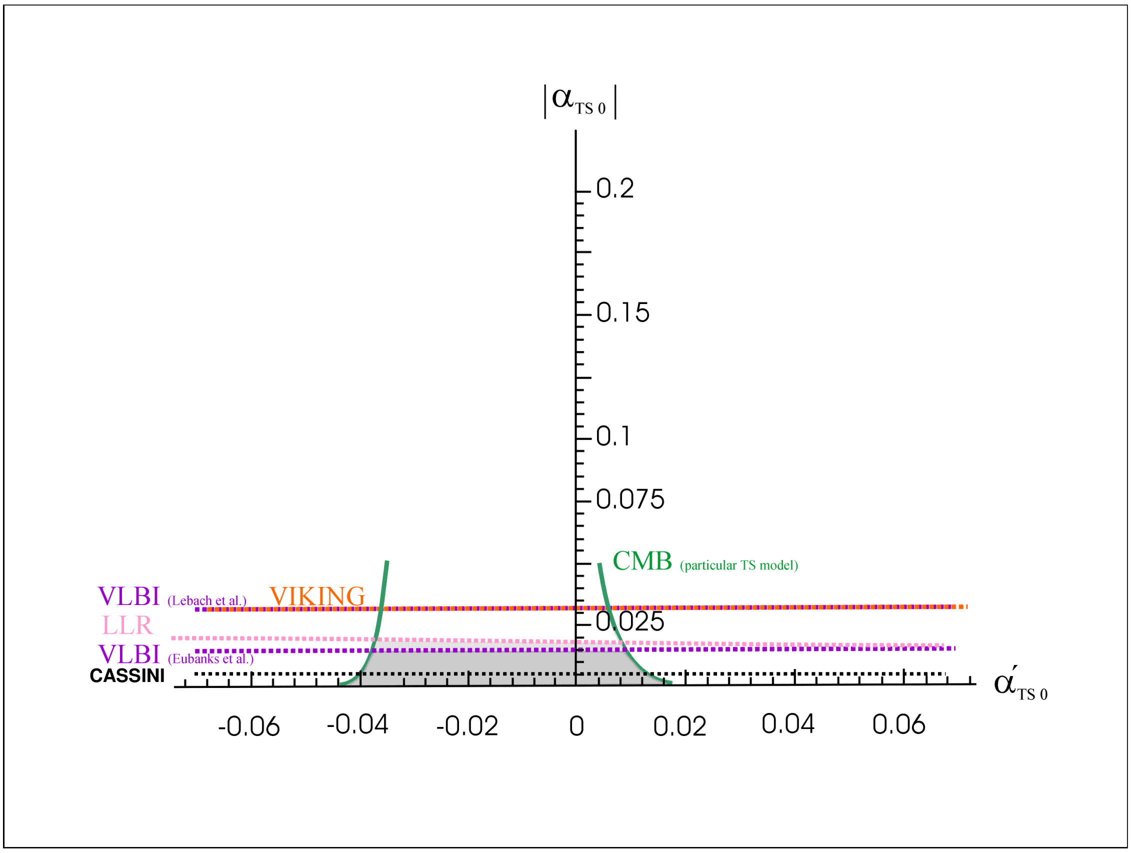
A **necessary condition** to recover GR from TS theories is to have an infinite value of parameter (null coupling ω_0). $\alpha_{TS\,0}$

Constraints on Tensor Scalar models from experiments:

In fact, only the square of $\alpha_{TS\,0}$ is physically measured.

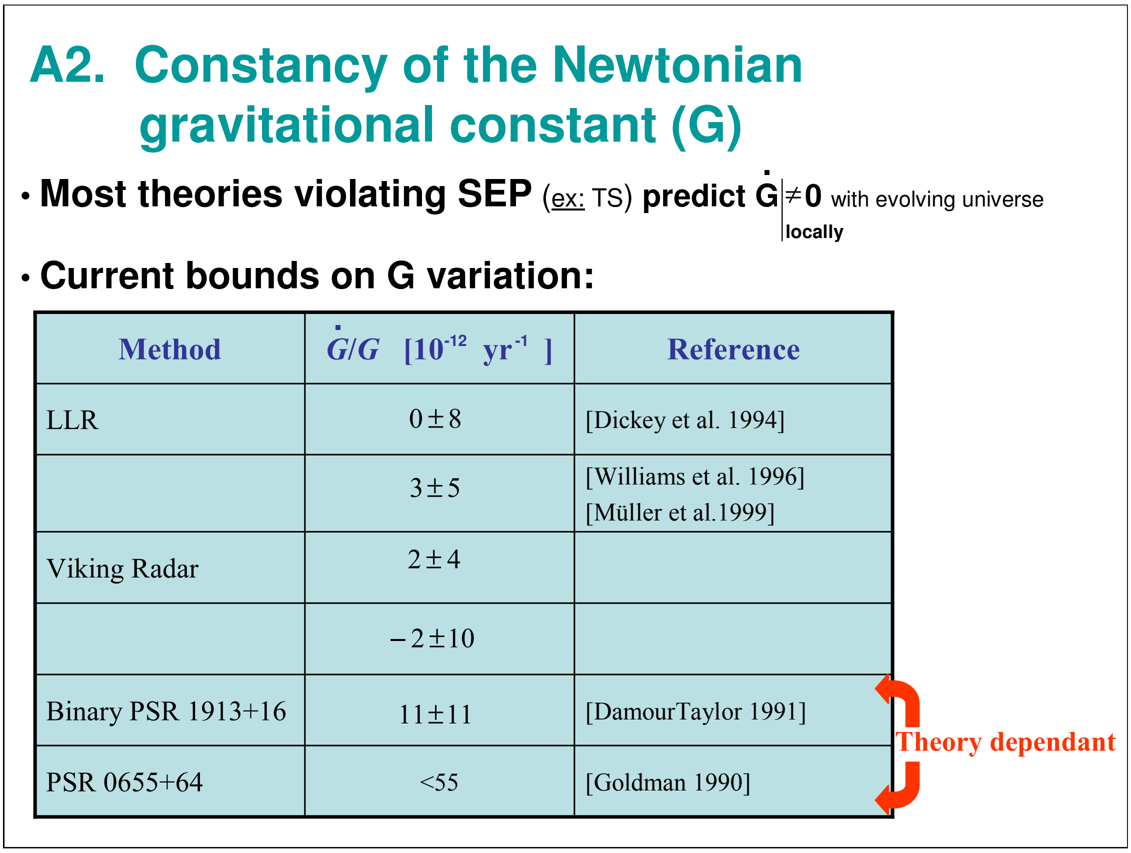
The regions allowed by solar system tests, respectively time-delay measurements to the Viking lander [Reasenberg et al. 1979] (VIKING, γ +1), Lunar Laser Ranging (LLR, η=4β - γ -3) [Williams et al. 2001], Mercury perihelion shift (λ = 2 + 2γ - β) , not apparent on this graph), Very Large Baseline Interferometry light deflection or time-delay measurements (VLBI, γ +1) [Eubanks et al. 1999], [Lebach et al. 1995], CASSINI Doppler Measurements [Bertotti et al. 2003] are beneath the corresponding dotted line. The regions allowed by binary pulsar tests, given by the dash lines [Damour et al. 1996a], [Damour et al. 1996b], [Damour et al. 1998], are respectively to the right of the corresponding curves for PSR 1913+16 and PSR 1534+12, while it is in between the corresponding dash line for PSR 0655+64. The parameter space allowed by pulsars strongly depends upon the equation of state adopted. Softer equations of state for nuclear matter lead to stronger constraints on the theory parameters. In this graph, a polytropic equation of state for the neutron stars was assumed. It can be seen through the plots presented in [Damour et al. 1998], how the constraints vary with the equation of state (Note that solar system tests in those graphs are a bit outdated). The shaded region represents the parameter space (,) allowed by solar system tests and binary pulsars simultaneously. Nevertheless, CASSINI data so far dominates constraints on and, for positive values of $\alpha_{TS\,0}$ $\alpha'_{TS\,0}$, solar sytem tests are still more efficient than pulsar data.

$\alpha_{TS\,0}$ $\alpha'_{TS\,0}$



Constraints on Tensor Scalar models from experiments:

This graph is to be compared with the previous figure. It shows how the constraints obtained by Chiba and his collaborators [Chiba et al. 1998] on the Tensor Scalar parameter space from the Cosmic Microwave Background (CMB) with spectral index n > 0.7 further shrunk drastically the $\alpha'_{TS\,0}$ -dimension. However, this CMB analysis was made solely for Damour et al.'s model.
The shaded region represents the parameter space ($\alpha_{TS\,0}$ $\alpha'_{TS\,0}$) allowed simultaneously by solar system tests and CMB data for this particular class of Tensor Scalar models. CMB constraints are about two orders of magnitude stronger than the pulsar or solar system limits on $\alpha_{TS\,0}$. But most recent VLBI or CASSINI data still provide the crucial information for the $\alpha_{TS\,0}$ -dimension.



A2. Constancy of the Newtonian gravitational constant (G)

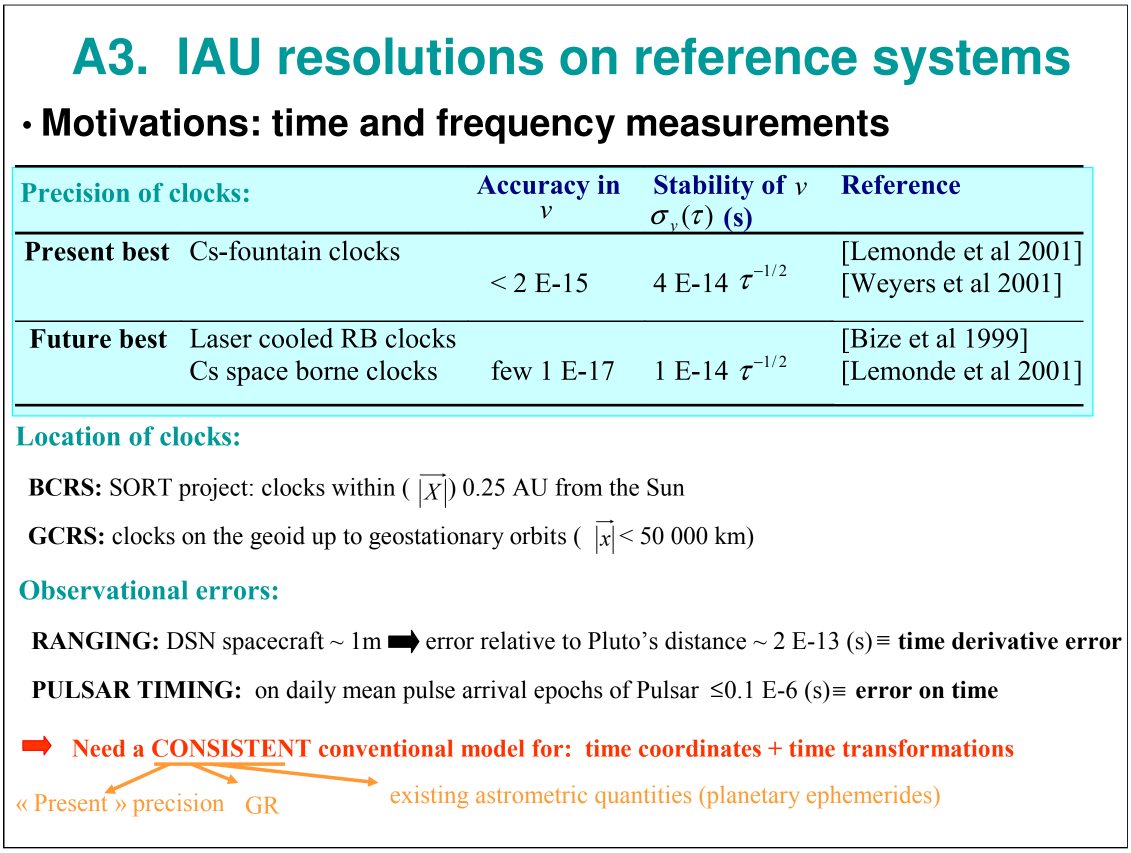
- **Most theories violating SEP** (ex: TS) **predict** $\dot{G}\big|_{\text{locally}} \neq 0$ with evolving universe
- **Current bounds on G variation:**

Method	$\dot{G}/G$ [10^{-12} yr^{-1}]	Reference
LLR	0 ± 8	[Dickey et al. 1994]
	3 ± 5	[Williams et al. 1996] [Müller et al.1999]
Viking Radar	2 ± 4	
	-2 ± 10	
Binary PSR 1913+16	11 ± 11	[DamourTaylor 1991]
PSR 0655+64	<55	[Goldman 1990]

Theory dependant

Previous methods to bound *G* variations: studies of Solar evolution, observations of Lunar occultations (including analysis of ancient eclipse data), LLR, planetary radar ranging measurements, pulsar timing data. [Will 1993]

Future methods to bound *G* variations: laboratory experiments (in a far future), Mercury orbiter over a 2-year mission (possibly down to $\Delta\left(\dot{G}/G\right) \approx 10^{-14}\ yr$ with 30cm accuracy in range).



A3. IAU resolutions on reference systems

- **Motivations: time and frequency measurements**

Precision of clocks:		**Accuracy in ν**	**Stability of ν $\sigma_y(\tau)$ (s)**	**Reference**
Present best	Cs-fountain clocks	< 2 E-15	4 E-14 $\tau^{-1/2}$	[Lemonde et al 2001] [Weyers et al 2001]
Future best	Laser cooled RB clocks Cs space borne clocks	few 1 E-17	1 E-14 $\tau^{-1/2}$	[Bize et al 1999] [Lemonde et al 2001]

Location of clocks:

BCRS: SORT project: clocks within ($|\vec{X}|$) 0.25 AU from the Sun

GCRS: clocks on the geoid up to geostationary orbits ($|\vec{x}| < 50\,000$ km)

Observational errors:

RANGING: DSN spacecraft ~ 1m ➡ error relative to Pluto's distance ~ 2 E-13 (s) ≡ **time derivative error**

PULSAR TIMING: on daily mean pulse arrival epochs of Pulsar ≤0.1 E-6 (s) ≡ **error on time**

➡ **Need a CONSISTENT conventional model for: time coordinates + time transformations**

« Present » precision GR existing astrometric quantities (planetary ephemerides)

The uncertainties allowed in the time transformations are < 5 E-18 in rate and < 0.2 E-12 s in amplitude for quasi periodic terms.

The spatial domain of validity of transformation has also to verify the present (future) observations available.

SORT = Solar Orbit Relativity Test.

$\sigma_y(\tau)$ (s) = Allan deviation. It provides a measure of the rms (residual mean squares) fractional frequency deviation among clocks, due to intrinsic noise processes in the clocks. Additional systematic effects (like frequency drifts or offsets must be accounted separately).

BCRS = Barycentric (centered on the Solar System barycenter) Celestial Reference System.
… used to model light propagation from distant celestial objects or the motion of bodies within the solar system (astrometry).
Hipparcos (…) catalog materialises it.

GCRS = Geocentric (centered on the Earth) Celestial Reference System.
… used to describe processes in the vicinity of Earth: Earth rotation, motion of Earth satellites… In analogy to GCRS a Planetary CRS can be constructed for any Planet in the solar System.

IAU2000 resolutions are based on the work of (Brumberg, Kopeikin, Klioner, Voinov) and (Damour, Soffel, Xu)

• IAU metric Gauge:

a) BCRS or GCRS coordinates:

Hyp: ➢ universe ⟨ homegeneous
isotropic

➢ isolated system: far outer regions - in free fall 1/1 cosmological model
- BCRS at rest 1/1 universe rest-frame

GCRS coord: comoving and kinematically non rotating X^{μ} 1/1 BCRS coord

- ➢ precession-nutation of E contained in model transforming ITRS to GCRS
- ➢ time variation of comological part of metric negligible 1/1 solar system exp.
- ➢ BCRS: $ds^2 = (\eta_{\mu\nu} + h_{\mu\nu}) \cdot dx^{\mu} \cdot dx^{\nu}$

  GCRS: $ds^2 = (\eta_{\mu\nu} + H_{\mu\nu}) \cdot dX^{\mu} \cdot dX^{\nu}$ with $H_{\mu\nu}$, $h_{\mu\nu}$ = PN-part >> cosmological effect
- ➢ x^{μ} asymptotically Minkowskian in outer regions

Universe rest-frame = frame in which universe is isotropic and homogeneous

ITRS = International Terrestrial Reference System (rotating with the Earth).

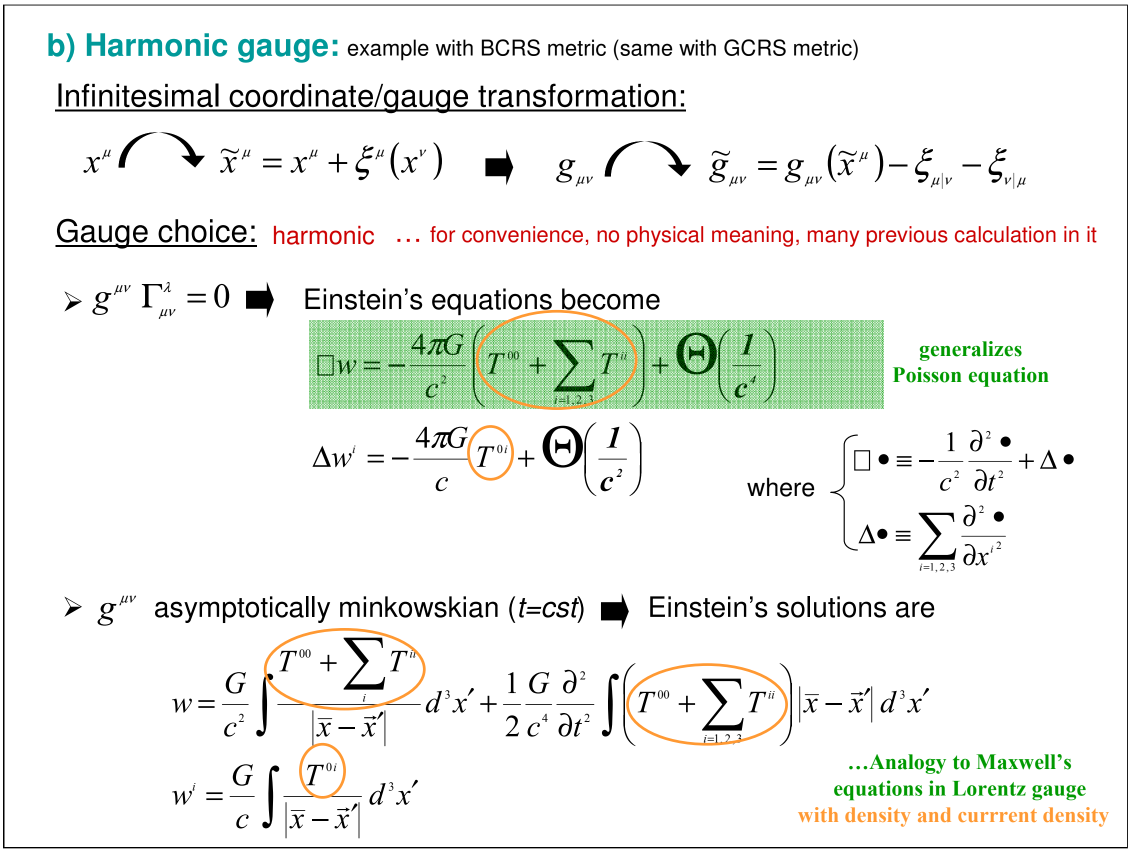
ex: geodesic precession ~ 1.9”/century
geodesic nutation ~ 0.15 marcsecond (amplitude of annual main term)
… see IAU precession-nutation model.

RW metric = Robertson-Walker metric
(associated with a cosmological evolutionary model of the Universe)

E = Earth

$\eta_{\mu\nu}$ = Minkowski metric

The GCRS frame is comoving, hence it does not follow a geodesic of the spacetime.



b) Harmonic gauge: example with BCRS metric (same with GCRS metric)

Infinitesimal coordinate/gauge transformation:

$$x^\mu \curvearrowright \tilde{x}^\mu = x^\mu + \xi^\mu(x^\nu) \quad \Rightarrow \quad g_{\mu\nu} \curvearrowright \tilde{g}_{\mu\nu} = g_{\mu\nu}(\tilde{x}^\mu) - \xi_{\mu|\nu} - \xi_{\nu|\mu}$$

Gauge choice: harmonic … for convenience, no physical meaning, many previous calculation in it

- $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$ ⇒ Einstein's equations become

$$\Box w = -\frac{4\pi G}{c^2}\left(T^{00} + \sum_{i=1,2,3} T^{ii}\right) + \Theta\left(\frac{1}{c^4}\right)$$

generalizes Poisson equation

$$\Delta w^i = -\frac{4\pi G}{c} T^{0i} + \Theta\left(\frac{1}{c^2}\right)$$

where $\begin{cases} \Box \bullet \equiv -\dfrac{1}{c^2}\dfrac{\partial^2 \bullet}{\partial t^2} + \Delta \bullet \\ \Delta \bullet \equiv \displaystyle\sum_{i=1,2,3} \dfrac{\partial^2 \bullet}{\partial x^{i\,2}} \end{cases}$

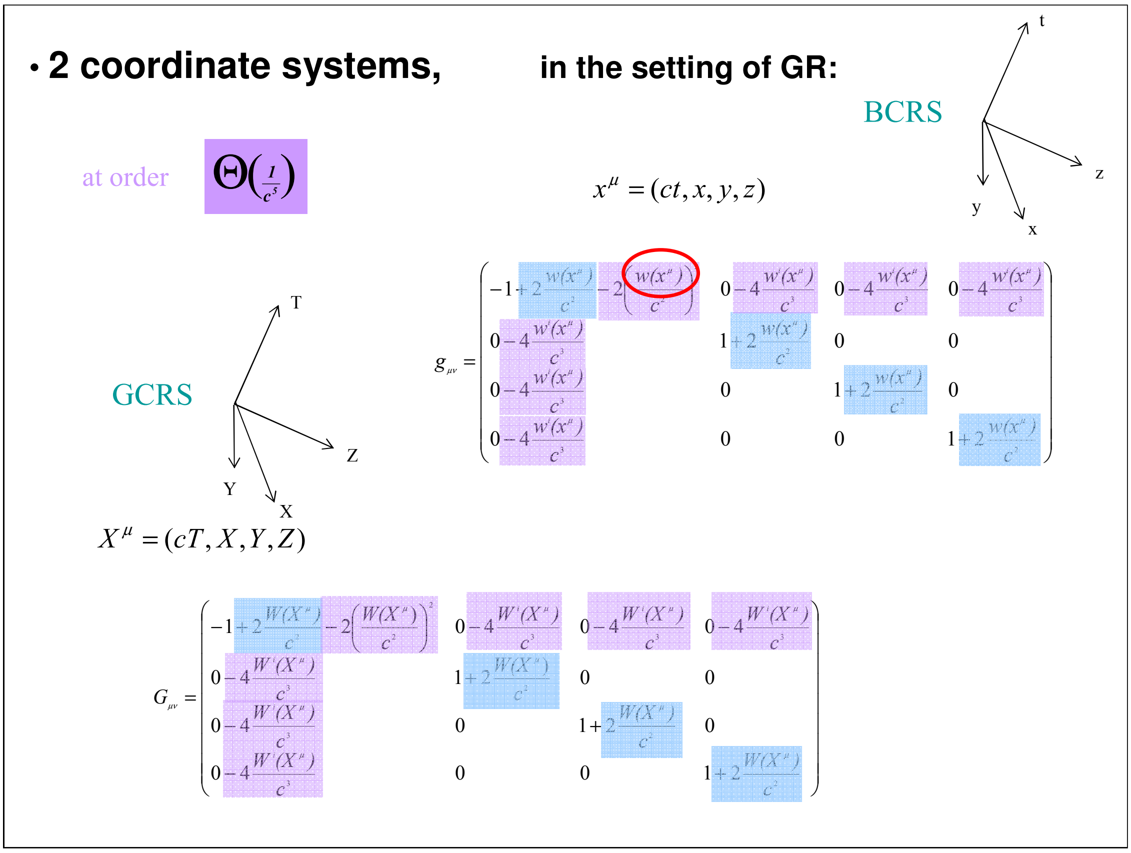
- $g^{\mu\nu}$ asymptotically minkowskian (*t=cst*) ⇒ Einstein's solutions are

$$w = \frac{G}{c^2}\int \frac{T^{00} + \sum_i T^{ii}}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{2}\frac{G}{c^4}\frac{\partial^2}{\partial t^2}\int \left(T^{00} + \sum_{i=1,2,3} T^{ii}\right)|\vec{x} - \vec{x}'|\, d^3x'$$

$$w^i = \frac{G}{c}\int \frac{T^{0i}}{|\vec{x} - \vec{x}'|} d^3x'$$

…Analogy to Maxwell's equations in Lorentz gauge with density and currrent density

SPN = Standard Post Newtonian gauge



- **2 coordinate systems,** **in the setting of GR:**

at order $\Theta(\frac{1}{c^5})$

BCRS

$$x^{\mu} = (ct, x, y, z)$$

$$g_{\mu\nu} = \begin{pmatrix} -1+2\frac{w(x^{\mu})}{c^2}-2\left(\frac{w(x^{\mu})}{c^2}\right) & 0-4\frac{w^i(x^{\mu})}{c^3} & 0-4\frac{w^i(x^{\mu})}{c^3} & 0-4\frac{w^i(x^{\mu})}{c^3} \\ 0-4\frac{w^i(x^{\mu})}{c^3} & 1+2\frac{w(x^{\mu})}{c^2} & 0 & 0 \\ 0-4\frac{w^i(x^{\mu})}{c^3} & 0 & 1+2\frac{w(x^{\mu})}{c^2} & 0 \\ 0-4\frac{w^i(x^{\mu})}{c^3} & 0 & 0 & 1+2\frac{w(x^{\mu})}{c^2} \end{pmatrix}$$

GCRS

$$X^{\mu} = (cT, X, Y, Z)$$

$$G_{\mu\nu} = \begin{pmatrix} -1+2\frac{W(X^{\mu})}{c^2}-2\left(\frac{W(X^{\mu})}{c^2}\right)^2 & 0-4\frac{W^i(X^{\mu})}{c^3} & 0-4\frac{W^i(X^{\mu})}{c^3} & 0-4\frac{W^i(X^{\mu})}{c^3} \\ 0-4\frac{W^i(X^{\mu})}{c^3} & 1+2\frac{W(X^{\mu})}{c^2} & 0 & 0 \\ 0-4\frac{W^i(X^{\mu})}{c^3} & 0 & 1+2\frac{W(X^{\mu})}{c^2} & 0 \\ 0-4\frac{W^i(X^{\mu})}{c^3} & 0 & 0 & 1+2\frac{W(X^{\mu})}{c^2} \end{pmatrix}$$

Small letters are used for the BCRS quantities while capital letters are used for GCRS quantities.

- **Link to SPN gauge?** In General Relativity ($\alpha = \beta = \gamma = 1$)…

using $\left\{ \begin{array}{l} \text{Einstein's solutions in harmonic gauge (see IAU2000)} \\ T^{\mu\nu} \text{definition (see PPN definitions)} \\ \text{PPN potential definitions} \end{array} \right.$

we find

$$w\Big|_{IAU\,2000} = \left(U + 2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4 + \frac{1}{2}\frac{\chi_{,tt}}{c^2} + \Theta\left(\frac{1}{c^4}\right) \right)_{\text{SPN}}$$

$$w^i\Big|_{IAU\,2000} = \left(V_i + \Theta\left(\frac{1}{c^2}\right) \right)_{\text{SPN}} \qquad \text{… to replace in } g^{\mu\nu}\Big|_{\text{harmonic of IAU2000}}$$

apply gauge transformation $\begin{pmatrix} w\big|_{SPN} = w\big|_{IAU\,2000} - \frac{1}{2}\frac{\chi_{,t}}{c^2} \\ w^i\big|_{SPN} = w^i\big|_{IAU\,2000} + \frac{1}{8}\frac{\chi_{,i}}{c^2} \end{pmatrix}$ with $\chi_{,ti} = \left(V_i - W_i\right)_{SPN}$

to recover $g^{\mu\nu}\Big|_{\text{SPN of PPN formalism}}$

SPN = Standard Post Newtonian gauge

Solutions to Einstein's equations in harmonic gauge:

$$w = \frac{G}{c^2}\int \frac{T^{00} + \sum_i T^{ii}}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{2}\frac{G}{c^4}\frac{\partial^2}{\partial t^2}\int \left(T^{00} + \sum_{i=1,2,3} T^{ii} \right) |\vec{x} - \vec{x}'|\, d^3x'$$

$$w^i = \frac{G}{c}\int \frac{T^{0i}}{|\vec{x} - \vec{x}'|} d^3x'$$

$T^{\mu\nu}$ **definition:**

$$T^{00} = \rho c^2 \left(1 + \frac{\Pi}{c^2} + \frac{v^2}{c^2} + 2\frac{U}{c^2} \right) + \Theta\left(\frac{1}{c^2}\right)$$

$$T^{0i} = \rho c\, v^i + \Theta\left(\frac{1}{c}\right)$$

$$T^{ij} = \rho\, v^i\, v^j + p\,\delta^{ij} + \Theta\left(\frac{1}{c^2}\right)$$

• Metric gravity potentials:

a) GCRS $X^\nu = (cTCG, X, Y, Z)$ $W^\mu(X^\nu) = (W, W^i)$

The scalar potential

$$W(X^\nu) \equiv W_E(X^\nu) + W_{ext\,E}(X^\nu)$$

Potential
at
clock

Potential
at
clock
due to Earth (E)

Potential
at
clock
due to other bodies

$$W_{0\,E}(X^\nu) + \overset{//}{W}_{L\,E}(X^\nu)$$

spherical Earth (degree 0)
- Earth mass monopole -

NON spherical Earth (degree L)
- Earth mass multipoles -

$$\overset{\backslash\backslash}{W}_{tidal\,E}(X^\nu) + W_{iner\,E}(X^\nu)$$

Earth tides
due to other solar
system bodies

GCRS-center
non inertial
**- Coupling higher order
multipole moments to external
gravitational fields -**

The vector potential (gravitomagnetic)

$$W^i(X^\nu) \equiv W^i{}_E(X^\nu) + W^i{}_{ext\,E}(X^\nu)$$

Potential
at
clock

Potential
at
clock
due to Earth (E)

Potential
at
clock
due to other bodies

$$W^i{}_{1\,E}(X^\nu) + \overset{//}{W}{}^i{}_{L\,E}(X^\nu) \qquad \overset{\backslash\backslash}{W}{}^i{}_{tidal\,E}(X^\nu) + W^i{}_{iner\,E}(X^\nu)$$

spherical Earth (degree 1)
- spin dipole -

rotating Earth (degree L)
- spin higher multipoles -

Earth tides
due to other solar
system bodies (E)

relativistic
Coriolis forces:
geodesic,
Lense-
Thirring,
Thomas
precessions

b) BCRS $x^\nu = (cTCB, x, y, z)$ $w^\mu(x^\nu) = (w, w^i)$

The scalar potential

$$w(x^\nu) \equiv \sum_{n=E,S,V,M\,....} w_n(x^\nu)$$

Potential
at
clock
due to body « n »

Potential
at
clock

$$w_{0\,n}(x^\nu) + w_{L\,n}(x^\nu) + \frac{\Delta_n(x^\nu)}{c^2}$$

spherical body « n »
(degree 0)
- « n » mass monopole -

coupling of the velocity,
acceleration or spin of body
« n » to the external
gravitational field
- « n » spin-orbit couplings -

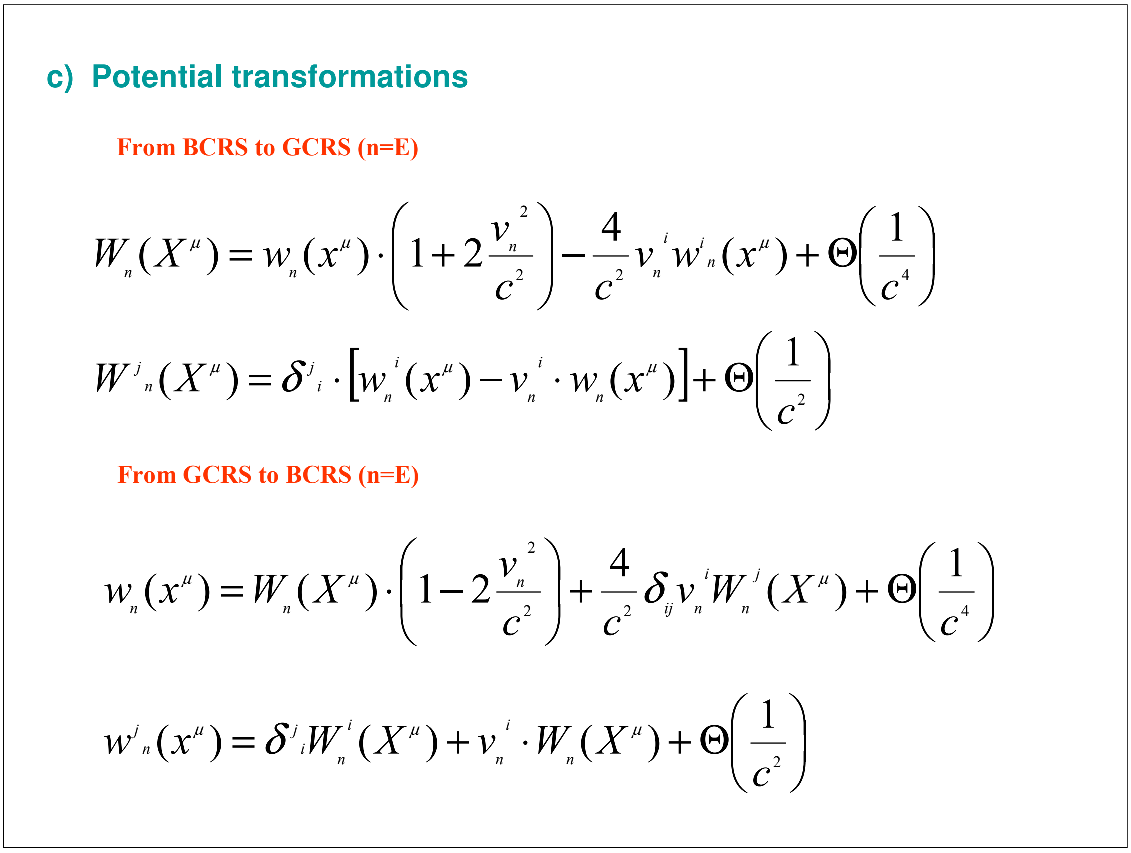
NON spherical « n » (degree L)
- « n » mass multipoles -

The vector potential (gravitomagentic)

$$w^i(x^\nu) \equiv \sum_{n=E,S,V,M\,....} w^i{}_n(x^\nu)$$

Potential
at
clock

Potential
at
clock
due to body « n »



c) Potential transformations

From BCRS to GCRS (n=E)

$$W_n(X^\mu) = w_n(x^\mu)\cdot\left(1+2\frac{v_n^2}{c^2}\right) - \frac{4}{c^2}v_n^i w^i{}_n(x^\mu) + \Theta\left(\frac{1}{c^4}\right)$$

$$W^j{}_n(X^\mu) = \delta^j{}_i \cdot \left[w_n^i(x^\mu) - v_n^i \cdot w_n(x^\mu)\right] + \Theta\left(\frac{1}{c^2}\right)$$

From GCRS to BCRS (n=E)

$$w_n(x^\mu) = W_n(X^\mu)\cdot\left(1-2\frac{v_n^2}{c^2}\right) + \frac{4}{c^2}\delta_{ij} v_n^i W_n^j(X^\mu) + \Theta\left(\frac{1}{c^4}\right)$$

$$w^j{}_n(x^\mu) = \delta^j{}_i W_n^i(X^\mu) + v_n^i \cdot W_n(X^\mu) + \Theta\left(\frac{1}{c^2}\right)$$

Potential transformations are obtained using the tensor transformation law

$$g_{\mu\nu}(t,\vec{x}) = G_{\alpha\beta}(T,\vec{X})\cdot\frac{\partial X^\alpha}{\partial x^\mu}\cdot\frac{\partial X^\beta}{\partial x^\nu}$$

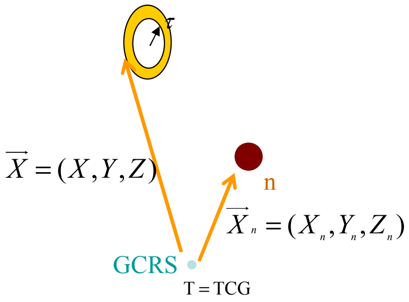
and by matching terms of equal power of c and for external potential terms, by matching terms of equal power w_i on both sides; while a specific shape for the coordinate transformation $X^\alpha(x^\mu)$ was sought.

Warning, this is a space-time (**4D**) transformation (generalized coordinate transformations)!!

d) Computation details in the GCRS

The Newtonian scalar potential on the clock due to body « n »

$$W_{0n}(X^{\mu}) \equiv \frac{\mathrm{GM_n}}{\mathrm{R_n}} \quad \text{with} \quad \vec{R}_n \equiv \vec{X} - \vec{X}_n$$



Higher order mass multipole contribution to the scalar potential on the clock due to body « n »

$$W_{L\,n} \cong \frac{GM_n}{|\vec{X}|}\sum_{l=1}^{l_{\max}}\sum_{m=0}^{l}\left(\frac{\Re_n}{|\vec{X}|}\right)^l \overline{P}_{lm}(\sin\varphi)\left[\overline{C}^n_{lm}\cos m\lambda + \overline{S}^n_{lm}\sin m\lambda\right]$$

… taken at order $l_{\max} = 0$ for all the Solar System bodies (mass monopole)
except from the one around which the satellite is orbiting (Earth or Mars, …)

where

(λ, φ) = longitude, latitude in *GCRS frame*

$\overline{P}_{lm}(\sin\varphi)$ = normalized Legendre polynomials

$(\overline{C}^n_{lm}, \overline{S}^n_{lm}) = \mathrm{fct}(T, \vec{X}) \approx \mathrm{fct}(\vec{X})$ normalized mass multipole coefficients of body "*n*" *in GCRS. Connected to Stokes coefficients from geopotential model by a time dependant transformation.*

$\Re_n$ = equatorial radius of "n"

GM_n = "gravitational mass" of "n" *in GCRS*

$W_{L\,n}$ is a scalar and can thus be calculated in any frame. Usually, it is calculated in the reference frame attached to body "*n*" (geopotential model). But the above expression is given in the GCRS frame.

The full Post-Newtonian calculation for mass multipole moments (Earth scalar potential coefficients) in terms of Blanchet-Damour moments is given in [Damour et al. 1991] (see also [Soffel et al. 2003]).

Tools for calculation:
Normalized Legendre polynomials versus Helmotz polynomials:

$$\overline{P}_{lm}(\sin\varphi) = \cos^m\varphi \cdot H_{lm}(\sin\varphi)$$

Normalized Legendre polynomials versus Legendre polynomials:

$$\overline{P}_{lm}(\sin\varphi) = \sqrt{\frac{(l+m)!}{(2-\delta_{lm})\cdot(2l+1)\cdot(l-m)!}}\,P_{lm}(\sin\varphi)$$

$$\overline{C}_{lm}\cdot\overline{P}_{lm}(\sin\varphi) = C_{lm}\cdot P_{lm}(\sin\varphi)$$

$$\overline{S}_{lm}\cdot\overline{P}_{lm}(\sin\varphi) = S_{lm}\cdot P_{lm}(\sin\varphi)$$

Recursive formulae for Helmotz polynomials:

for l=m

$$H_{00} = 1$$

$$H_{11} = \sqrt{3}$$

$$H_{m,m} = \alpha_{m,m}\cdot H_{m-1,m-1}$$

$$\alpha_{m,m} = \sqrt{1+\frac{1}{2m}}$$

for l=m+1

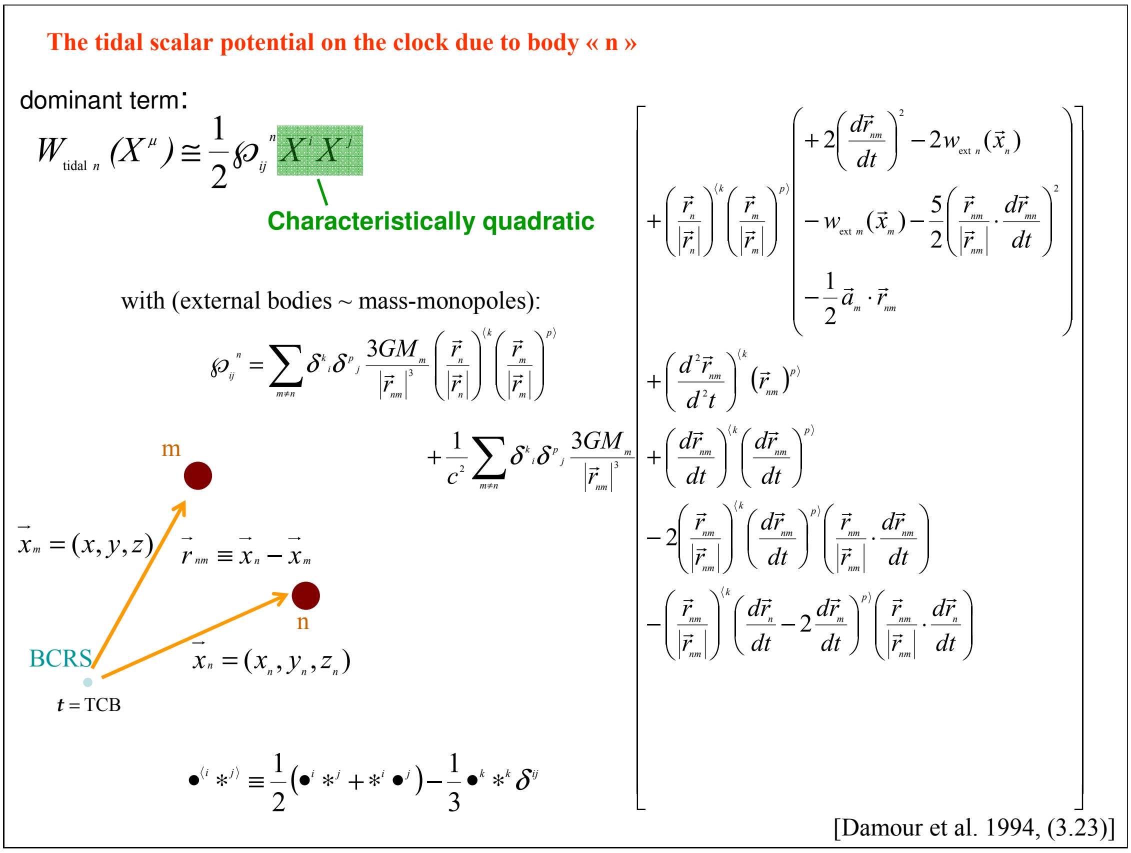
$$H_{m+1,m} = \alpha_{l,m}\cdot H_{m,m}\cdot\sin\varphi$$

$$\alpha_{m+1,m} = \sqrt{2m+3}$$

for l>m+1

$$H_{l,m} = \alpha_{l,m}\cdot\left[\sin\varphi\cdot H_{l-1,m} - \frac{H_{l-2,m}}{\alpha_{l-1,m}}\right]$$

$$\alpha_{l,m} = \sqrt{\frac{(2l+1)(2l-1)}{(l-m)(l+m)}}$$



The tidal scalar potential on the clock due to body « n »

dominant term:

$$W_{\text{tidal }n}(X^{\mu}) \cong \frac{1}{2}\wp_{ij}^{\ n} X^i X^j$$

Characteristically quadratic

with (external bodies ~ mass-monopoles):

$$\wp_{ij}^{\ n} = \sum_{m\neq n} \delta^k{}_i \delta^p{}_j \frac{3GM_m}{|\vec{r}_{nm}|^3}\left(\frac{\vec{r}_n}{|\vec{r}_n|}\right)^{\langle k}\left(\frac{\vec{r}_m}{|\vec{r}_m|}\right)^{p\rangle}$$

$$+\frac{1}{c^2}\sum_{m\neq n}\delta^k{}_i\delta^p{}_j\frac{3GM_m}{|\vec{r}_{nm}|^3}\left[\begin{array}{l} +\left(\frac{\vec{r}_n}{|\vec{r}_n|}\right)^{\langle k}\left(\frac{\vec{r}_m}{|\vec{r}_m|}\right)^{p\rangle}\left(\begin{array}{l} +2\left(\frac{d\vec{r}_{nm}}{dt}\right)^2-2w_{\text{ext }n}(\vec{x}_n) \\ -w_{\text{ext }m}(\vec{x}_m)-\frac{5}{2}\left(\frac{\vec{r}_{nm}}{|\vec{r}_{nm}|}\cdot\frac{d\vec{r}_{mn}}{dt}\right)^2 \\ -\frac{1}{2}\vec{a}_m\cdot\vec{r}_{nm} \end{array}\right) \\ +\left(\frac{d^2\vec{r}_{nm}}{d^2t}\right)^{\langle k}(\vec{r}_{nm})^{p\rangle} \\ +\left(\frac{d\vec{r}_{nm}}{dt}\right)^{\langle k}\left(\frac{d\vec{r}_{nm}}{dt}\right)^{p\rangle} \\ -2\left(\frac{\vec{r}_{nm}}{|\vec{r}_{nm}|}\right)^{\langle k}\left(\frac{d\vec{r}_{nm}}{dt}\right)^{p\rangle}\left(\frac{\vec{r}_{nm}}{|\vec{r}_{nm}|}\cdot\frac{d\vec{r}_{nm}}{dt}\right) \\ -\left(\frac{\vec{r}_{nm}}{|\vec{r}_{nm}|}\right)^{\langle k}\left(\frac{d\vec{r}_n}{dt}-2\frac{d\vec{r}_m}{dt}\right)^{p\rangle}\left(\frac{\vec{r}_{nm}}{|\vec{r}_{nm}|}\cdot\frac{d\vec{r}_n}{dt}\right) \end{array}\right]$$

$\vec{x}_m = (x, y, z)$

$\vec{r}_{nm} \equiv \vec{x}_n - \vec{x}_m$

$\vec{x}_n = (x_n, y_n, z_n)$

$$\bullet^{\langle i} *^{j\rangle} \equiv \frac{1}{2}(\bullet^i *^j + *^i \bullet^j) - \frac{1}{3}\bullet^k *^k \delta^{ij}$$

[Damour et al. 1994, (3.23)]

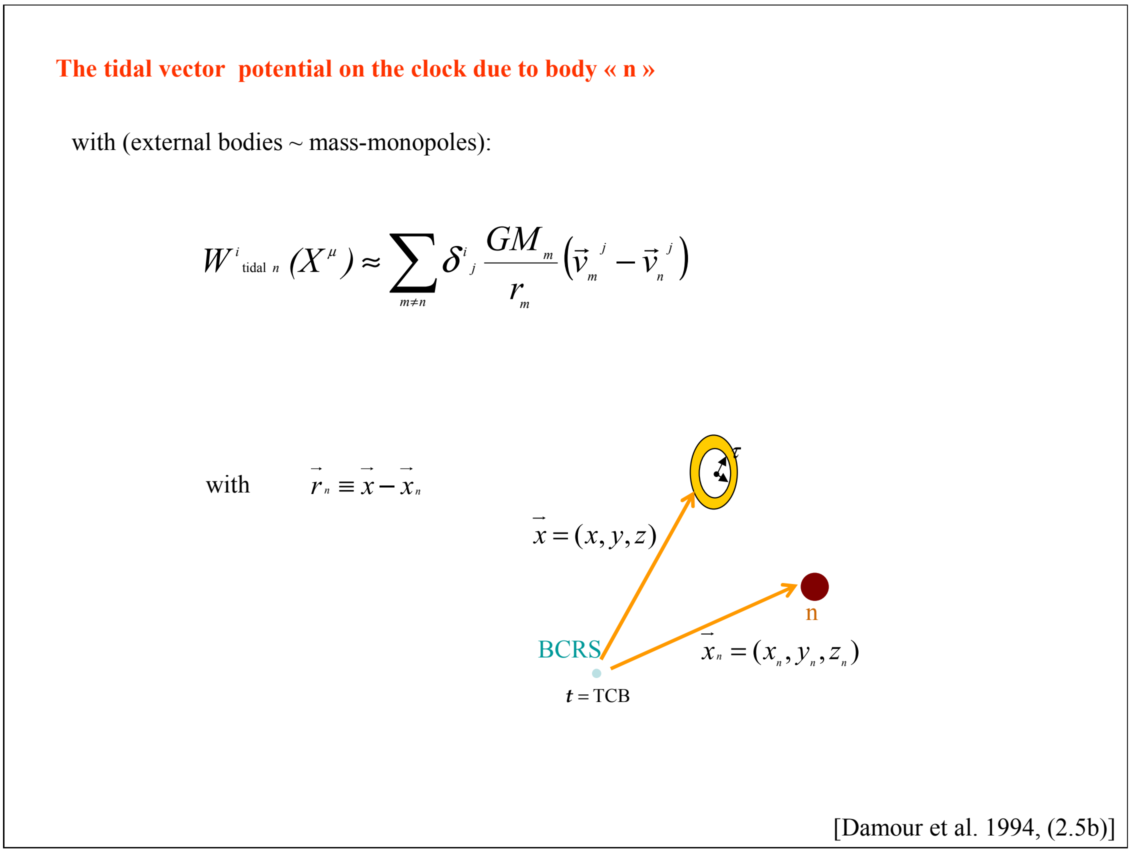
Linear terms representing inertial forces may also exist, but can be eliminated by a suitable choice of the origin of local coordinates.

In the Newtonian limit, for the Earth (GCRS):

$$W_{\text{tidal }E}(T,\vec{X}) \cong w_{ext\,\mathrm{E}}(\vec{x}_E + \vec{X}) - w_{ext\,\mathrm{E}}(\vec{x}_E) - \vec{X}\cdot\vec{\nabla}w_{ext\,\mathrm{E}}(\vec{x}_E)$$

Full Post-Newtonian expressions:

- [Damour et al. 1992] by means of tidal moments, with $W_{ext\,\mathrm{E}}$ written as $\overline{W}$
- [Klioner et al. 1993] in a closed form
- [Klioner et al. 2000] for higer order terms than presented in this transparent



The tidal vector potential on the clock due to body « n »

with (external bodies ~ mass-monopoles):

$$W^i_{\text{tidal } n}(X^\mu) \approx \sum_{m\neq n} \delta^i_j \frac{GM_m}{r_m}\left(\vec{v}_m^{\,j} - \vec{v}_n^{\,j}\right)$$

with $\vec{r}_n \equiv \vec{x} - \vec{x}_n$

[Damour et al. 1994, (2.5b)]

Full Post-Newtonian expressions:

- [Damour et al. 1992] by means of tidal moments, with $W_{ext\,\mathrm{E}}$ written as $\overline{W}$
- [Klioner et al. 1993] in a closed form
- [Klioner et al. 2000] for higer order terms than presented in this transparent

The inertial scalar potential on the clock due to body « n »

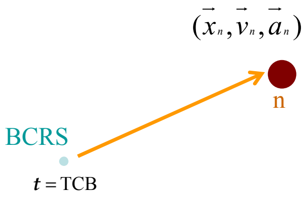
Characteristically linear

$$W_{iner\ n}(X^{\mu}) = Q_{n\ i} \cdot X^{i} \quad \text{with} \quad Q^{i}_{\ n} = \delta^{i}_{\ j}\left[\frac{\partial w_{ext\,n}}{\partial x^{j}}\left(x_{n}^{\ \mu}\right) - a_{n}^{\ j}\right] + \Theta\left(\frac{1}{c^{2}}\right)$$

acceleration of geocenter 1/1 geodesic motion

$$W^{i}_{\ iner\ n}(X^{\mu}) = -\frac{1}{4}c^{2}\varepsilon_{ijk}\Omega^{j}_{\ iner\ n}X^{k} \quad \text{with} \quad \vec{\Omega}_{iner\ n} = \vec{\Omega}_{GP\ n} + \vec{\Omega}_{LTP\ n} + \vec{\Omega}_{TP\ n}$$

Relativistic Coriolis



$$\vec{\Omega}_{GP\ n} = -\frac{3}{2c^{2}}\vec{v}_{n} \times \vec{\nabla} w_{ext\,n}\left(x_{n}^{\ \mu}\right)$$

Geodesic Precession

$$\vec{\Omega}_{LTP\ n} = +\frac{2}{c^{2}}\vec{\nabla} \times \vec{w}_{ext\,n}\left(x_{n}^{\ \mu}\right)$$

Lense-Thirring Precession

$$\vec{\Omega}_{TP\ n} = -\frac{1}{2c^{2}}\vec{v}_{n} \times \vec{Q}_{n}$$

Thomas Precession

The vector potential on the clock due to central body « n »

IF "n" is spherically symmetric:

$$W^{i}_{\ n}(X^{\mu}) = \frac{G}{2}\frac{\left(\vec{S}_{n} \times \vec{X}\right)^{i}}{\left|\vec{X}\right|^{3}}$$

…approximation for every Solar System body with intrinsic spin $\vec{S}_{n}$

$Q^{i}_{\ n}$ vanishes for a non-rotating spherical body *n* (mass monopole only).
The full Post-Newtonian expression can be derived from (6.30a) in Damour-Soffel-Xu (with *Q* written as *G*) [Damour et al. 1991] .
ex: contribution of the Moon to $Q^{i}_{\ E}$4 E-11 m/s²

$\Omega_{GP\ E}$ ~ 2 "/century … mostly due to the Sun static gravitational field

$\Omega_{LTP\ E}$ ~ 2 E-3 "/century … mostly due to the motion and spin of Moon and Sun

$\Omega_{TP\ E}$ ~ 4 E-9 "/century … negligible 1/1 geodesic precession

➡ $\Omega_{iner\ E} \sim \Omega_{GP\ E}$ = precession of the Earth in the GCRS
… it is the precession of locally inertial coordinates with respect to GRGS. GCRS is kinematically non-rotating 1/1 BCRS. GCRS is not a locally inertial reference system, and thus Coriolis forces caused by geodesic precession-nutation appear in every GCRS dynamical equations of motion (ex: Earth satellites).

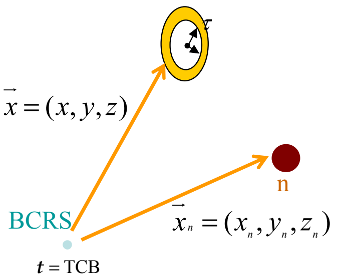
The full Post-Newtonian calculation for spin multipole moments (Earth vector potential coefficients) should be made in terms of Blanchet-Damour moments: to write $W_{i\,n}$ without approximating "*n*" as a spherically symmetric body, one has to use $W_{i\,L\,n}$. But for a post-Newtonian metric, it is sufficient to use the Newtonian rigid rotation model because $W_{i\,n}$ only plays a role in very small relativistic effects (GP, LTP, TP and time transformations). [Damour et al. 1991] (see also [Soffel et al. 2003])

e) Computation details in the BCRS

The Newtonian scalar potential on the clock due to body « n »

$$w_{0n}(x^\mu) \equiv \frac{\mathrm{GM}_\mathrm{n}}{\mathrm{r}_\mathrm{n}} \quad \text{with} \quad \vec{r}_n \equiv \vec{x} - \vec{x}_n$$

$$w_{ext\ n}(x^\mu) \equiv \sum_{m \neq n} w_m(x^\mu)$$



Higher order mass multipole contribution to the scalar potential on the clock due to body « n »

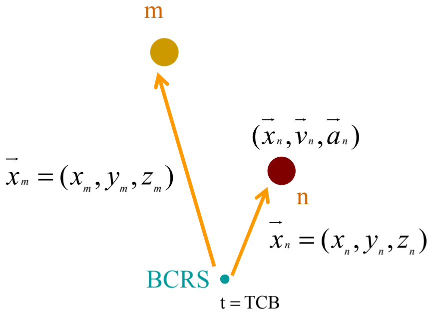
$w_{Ln}(x^\mu)$ … taken as $= 0$ for all the Solar System bodies (mass monopole) except from the one around which the satellite is orbiting (Earth or Mars, …).

… calculated from $W_{Ln}(X^\mu)$ with the potential tranformation rule

The coupling of the velocity, acceleration or spin of body « n » to external gravitational fields felt by the clock

$$\Delta_n(x^\mu) \cong \frac{GM_n}{r_n}\left[\left(-2v^2_n + \sum_{m\neq n}\frac{GM_m}{r_{mn}}\right) + \frac{1}{2}\left(\frac{(\vec{r}_n\cdot\vec{v}_n)^2}{r_n^2} + r_n^k a_n^k\right)\right] + \frac{2G\vec{v}_n\cdot(\vec{r}_n\times\vec{S}_n)}{r_n^3}$$

with $\vec{r}_{mn} \equiv \vec{x}_m - \vec{x}_n$



The vector potential on the clock due to body « n »

$w^i_n(x^\mu)$… calculated from $W^i_n(X^\mu)$ with the potential transformation rule:

$$\cong \frac{G}{2}\frac{(\vec{S}_n\times\vec{r}_n)^i}{|\vec{r}_n|^3} + G\frac{M_n v_n^i}{|\vec{r}_n|}$$

, $\vec{S}_n\cdot\vec{v}_n$ term neglected

…with $\vec{S}_n \sim 0$ for all the Solar System bodies (mass monopole)
except from the one around which the satellite is orbiting (Earth or Mars, …).

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