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A general relativistic methodology for laser links. Illustration with LISA.

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OUTLINE OF THE SPEACH

Part A- CONTEXT: general relativity in geodesy

1. **GR** (General Relativity) **IN THE BIG PICTURE**
2. **GR IN REFERENCE FRAMES AND SPACE-TIME TRANSFORMATIONS**
3. **GR IN CLOCK FREQUENCIES AND TIME**
4. **GR IN LASER LINKS** — this speach
5. **GR IN ORBITOGRAPHY** — previous speach
6. **MOTIVATION FOR A NAIVE GR APPROACH VIA THE METRIC**

CCT ORB January 2004

Part B- GR METHODOLOGY FOR LASER LINKS

1. **LASER LINK PICTURE**
2. **PRINCIPLE OF PHOTON FLIGHT TIME**
3. **PRINCIPLE OF FREQUENCY SHIFT**

Part C- GR LASER LINKS IN LISA

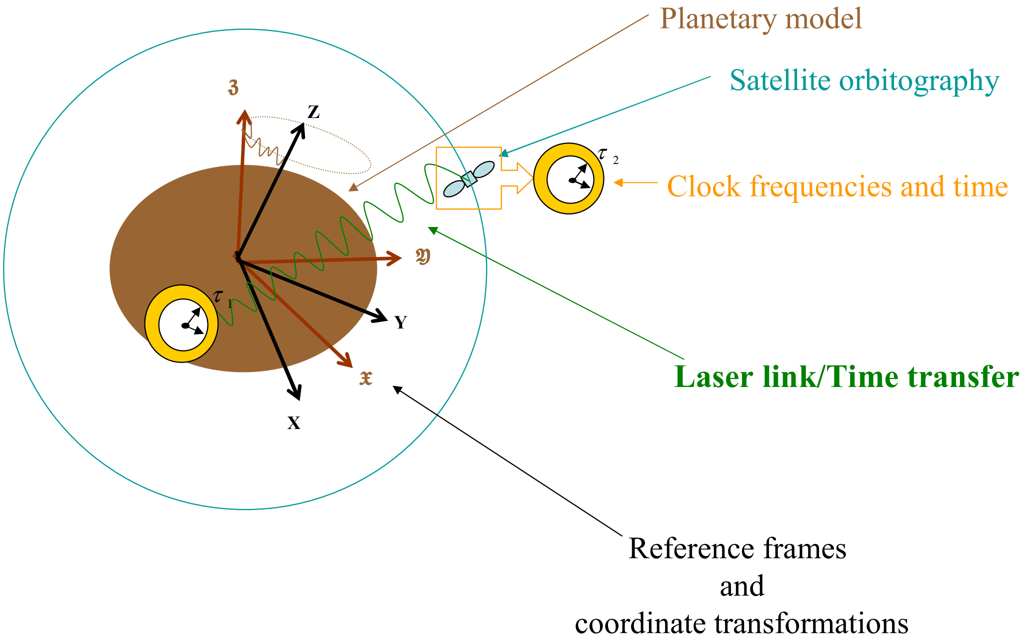
1. FACTS ABOUT LISA'S ORBITOGRAPHY AND LASER LINKS
2. PRINCIPLE: GR laser link applied to LISA
3. LISA FLIGHT TIME SOLUTION
4. LISA FREQUENCY SHIFT SOLUTION

Part D- CONCLUSION

Part A- CONTEXT:
general relativity in geodesy

A1. GR IN THE BIG PICTURE

Precise geophysics requires precise geodesy:



GR in space geodesy: GR comes into play at every steps of the processes described here:

- In **space geodesy**, the aim is to obtain information on the planet, to build a **planetary model**,

- thanks to the knowledge of the **orbit** of a satellite around it.

- To gain such knowledge, one needs to know very precisely the position of the satellite as a function of time with respect to the crust **reference frame**.

- But since the planet rotates, this frame is related to a quasi inertial frame in a non trivial way (rotation and nutation).

- To compute the position of the satellite (orbitography reduction), a **laser link** is used. It consists in computing the time transfer of photons between a receiver on board and a emitter on ground (or vice versa).

- This requires a clock on ground and another one on board of the satellite, each beating their own **time and frequency**, to monitor the emission time and the arrival time of photons.

A2. MOTIVATION FOR A NATIVE GR APPROACH

Relevance of the metric in GR:

Given the coordinates $x^{\mu} = (ct, x, y, z)$ **or alternatively** $X^{\mu} = (cT, X, Y, Z)$

$$c^2\, d\tau^2 = ds^2 = g_{\mu\nu}(x^{\mu}) \cdot dx^{\mu} \cdot dx^{\nu} = G_{\mu\nu}(X^{\mu}) \cdot dX^{\mu} \cdot dX^{\nu}$$

proper time

Invariant line-element

Spacetime coordinates

metric…
describing space-time geometry

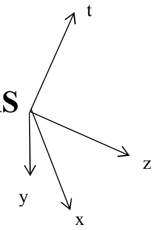
- **Relation between** τ **and** t **: relativistic time dilation integral**
- **Relation between** t **and** (x,y,z) **: (Geodesic) equations of motion (if only gravitational forces)**
- **Relation between** x^{μ} **and** X^{μ}**: coordinate space-time transformations**

Use a native coherent GR approach based on metric … instead of classic + corrections

2 coordinate systems, in the setting of GR:

at order $\Theta\left(\frac{1}{c^5}\right)$

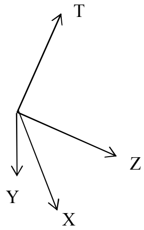
BCRS



$$x^{\mu} = (ct, x, y, z)$$

$$g_{\mu\nu} = \begin{pmatrix} -1+2\frac{w(x^{\mu})}{c^2}-2\left(\frac{w(x^{\mu})}{c^2}\right) & 0-4\frac{w'(x^{\mu})}{c^3} & 0-4\frac{w'(x^{\mu})}{c^3} & 0-4\frac{w'(x^{\mu})}{c^3} \\ 0-4\frac{w'(x^{\mu})}{c^3} & 1+2\frac{w(x^{\mu})}{c^2} & 0 & 0 \\ 0-4\frac{w'(x^{\mu})}{c^3} & 0 & 1+2\frac{w(x^{\mu})}{c^2} & 0 \\ 0-4\frac{w'(x^{\mu})}{c^3} & 0 & 0 & 1+2\frac{w(x^{\mu})}{c^2} \end{pmatrix}$$

GCRS



$$X^{\mu} = (cT, X, Y, Z)$$

$$G_{\mu\nu} = \begin{pmatrix} -1+2\frac{W(X^{\mu})}{c^2}-2\left(\frac{W(X^{\mu})}{c^2}\right)^2 & 0-4\frac{W'(X^{\mu})}{c^3} & 0-4\frac{W'(X^{\mu})}{c^3} & 0-4\frac{W'(X^{\mu})}{c^3} \\ 0-4\frac{W'(X^{\mu})}{c^3} & 1+2\frac{W(X^{\mu})}{c^2} & 0 & 0 \\ 0-4\frac{W'(X^{\mu})}{c^3} & 0 & 1+2\frac{W(X^{\mu})}{c^2} & 0 \\ 0-4\frac{W'(X^{\mu})}{c^3} & 0 & 0 & 1+2\frac{W(X^{\mu})}{c^2} \end{pmatrix}$$

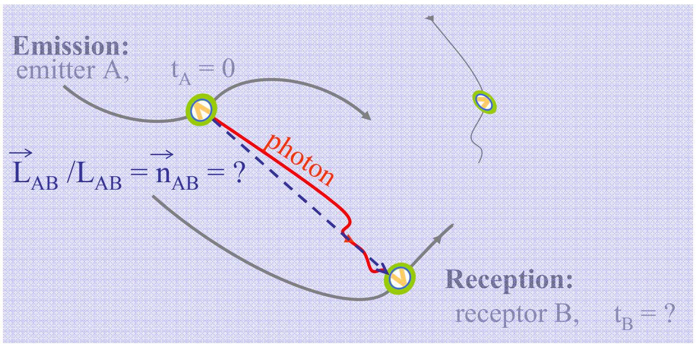
IAU = International Astronomical Union

BCRS = Barycentric Coordinate Reference System

GCRS = Geocentric Coordinate Reference System

Part B- GR METHODOLOGY FOR LASER LINKS

B1. LASER LINK PICTURE



- **Chose the appropriate background metric:**

$$ds^2 = g_{\mu\nu}(x^\mu) \cdot dx^\mu \cdot dx^\nu$$

<u>ex</u>: motion around Sun (BCRS) spherically symmetric, without planets, up to 1st order

$$ds^2 = -\left[1 - \frac{2GM}{r\,c^2} + O\left(\left(\frac{GM}{r\,c^2}\right)^2\right)\right] c^2 dt^2 + \left[1 + \frac{2GM}{r\,c^2} + O\left(\left(\frac{GM}{r\,c^2}\right)^2\right)\right] \left(dx^2 + dy^2 + dz^2\right)$$

B2. PRINCIPLE OF PHOTON FLIGHT TIME

$$\begin{cases} \dfrac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma + K_\beta \left(g^{\alpha\beta} - u^\alpha u^\beta\right) \\ u^\alpha = \dfrac{dx^\alpha}{d\tau} \text{ and first integral } g_{\alpha\beta} u^\alpha u^\beta = c^2 \end{cases} \text{ with } \begin{cases} K_\alpha \equiv \text{quadri-"force"}(=0 \text{ for photons}) \\ x^\alpha = (ct, x, y, z) \\ \alpha, \beta, \gamma = 0, 1, 2, 3 \\ \tau = \text{proper time (replaced by } \lambda \text{ for photons)} \\ \Gamma^\alpha_{\beta\gamma} = \text{Christoffel symbol } wrt \text{ metric } g_{\alpha\beta} \end{cases}$$

- **Matching of photon orbit (geodesic) with emitter orbit at emission time t_A:**

$$\vec{x}_{ph}\left(t_A, \vec{n}_{AB}(t_A), \vec{x}_A(t_A)\right) = \vec{x}_A\left(t_A, \vec{x}_A(t_A), \vec{v}_A(t_A)\right)$$

Photon geodesic orbit — Emitter orbit

- **Matching of photon orbit (geodesic) with receptor orbit at emission time t_B:**

$$\vec{x}_{ph}\left(t_B, \vec{n}_{AB}(t_B), \vec{x}_A(t_B)\right) = \vec{x}_B\left(t_B, \vec{x}_B(t_B), \vec{v}_B(t_B)\right)$$

Photon geodesic orbit — Receptor orbit

- **Equation to be solved in terms of quantities at emission time t_A:**

➡ $\left(t_{AB} = t_B - t_A, \vec{n}_{AB}\right)$ = (flight time, « direction ») = 1 + 2 (normalization) = 3 unknowns

We have got three equations with three unknowns

B3. PRINCIPLE OF FREQUENCY SHIFT

- **Energy measured from emitter A or receptor B:**

$$E_{A\,or\,B} = -g_{\alpha\beta}\,.u^{\alpha}_{ph}\,.u^{\beta}_{A\,or\,B} \quad \text{where} \quad u^{\alpha}_{A\,or\,B} = dx^{\alpha}_{A\,or\,B}/d\tau_{A\,or\,B}$$ = 4-proper-velocity of A or B

$$v^{i}_{A\,or\,B} = dx^{i}_{A\,or\,B}/dt$$ = 3-velocity of A or B

$$u^{\alpha}_{ph} = dx^{\alpha}_{ph}/d\lambda$$ = photon 4-wave vector

- **Frequency shift =** $z_{AB}(t_A) = \dfrac{E_B(t_B = t_A + t_{AB})}{E_A(t_A)} - 1$

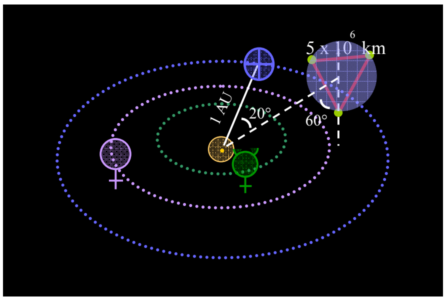
= relative difference between (if transfer from A to B)

- frequency of photon, emitted by A, measured when received at B
- proper frequency of photon when emitted by A
(= proper frequency of identical oscillators aboard A and B)

Part C- GR LASER LINKS IN LISA

C1. FACTS ABOUT LISA'S

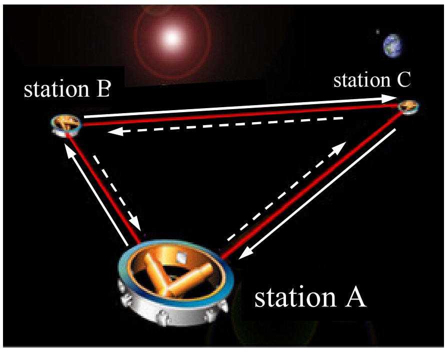
Orbitography:



coordinate position/velocity $(\vec{x}_A, \vec{v}_A, \vec{x}_B, \vec{v}_B, \vec{x}_C, \vec{v}_C)_t$?

inter-distance $L_{AB}(t)$?

Laser link:



photon propagation time $t_{AB}(t)$?

frequency shift $z_{AB}(t)$?

C2. PRINCIPLE: GR laser link applied to LISA

FOR LISA:

- emitter = spacecraft

- receptor = spacecraft

- assume only one test mass per spacecraft,
no non gravitational force,
➡ K_α=0 in GR equations of motion

- **use BCRS IAU2000 metric in GR laser link method**

- assume a spherical non-rotating Sun

- assume no planets

- **assume identical ideal clocks aboard spacecrafts**

C3. LISA FLIGHT TIME SOLUTION

[Chauvineau, Pireaux, Regimbau, Vinet, Phys. Rev. D 2005]

$$t_{AB} = \overset{0}{t}_{AB} + \overset{1/2}{t}_{AB} + \overset{1}{t}_{AB} + O\left(\varepsilon^{3/2}\right)$$

- **order 0 :** $\overset{0}{t}_{AB} = \frac{L_{AB0}}{c}$ where $L_{AB0} = \pm\left|\vec{x}_{B0} - \vec{x}_{A0}\right|$ (+ sign : photon travels from A to B)

  Classical — evaluated at t_A

- **order 1/2 :** $\overset{1/2}{t}_{AB} = \overset{0}{t}\,\frac{\overset{0}{\vec{n}}_{AB}\cdot\vec{v}_{B0}}{c}$ where $\overset{0}{\vec{n}}_{AB} = \frac{\vec{x}_{B0} - \vec{x}_{A0}}{L_{AB0}}$

  Kinematic terms (Sagnac and aberration)

- **order 1 :** $\overset{1}{t}_{AB} = \frac{1}{2}\overset{0}{t}\left[\frac{v_{B0}^2}{c^2} + \left(\frac{\overset{0}{\vec{n}}_{AB}\cdot\vec{v}_{B0}}{c}\right)^2\right] + \overset{0}{\vec{n}}_{AB}\left[(1+\gamma)\frac{GM}{c^3}\vec{\chi}\left(\overset{0}{t}, \overset{0}{\vec{n}}_{AB}\right) - \frac{GM\,\vec{x}_{B0}}{2\,r_{B0}^3\,c}\overset{0}{t}^2\right]$

  Kinematic terms — Shapiro delay — Velocity change during photon flight time

where $\vec{\chi}\left(t, \vec{n}_{AB}\right) = (r - r_0)\vec{P} + \vec{n}_{AB}\ln\frac{\vec{n}_{AB}\,.\,\vec{x}_{A0} + c\,t + r}{\vec{n}_{AB}\,.\,\vec{x}_{A0} + r_0}$ and $\vec{P} = \frac{\vec{x}_{A0} - \left(\vec{n}_{AB}\,.\,\vec{x}_{A0}\right)\vec{n}_{AB}}{r_{A0}^2 - \left(\vec{n}_{AB}\,.\,\vec{x}_{A0}\right)^2}$

A l’ordre 0, le temps de parcours est l’inter-distance AB à l’émission, divisée par la vitesse de la lumière.

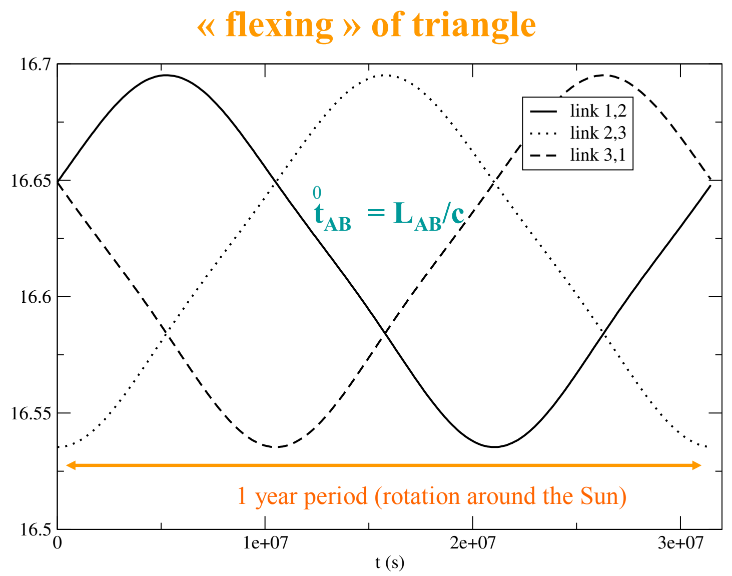
A l’ordre ½, on a un doppler classique dû au mouvement des stations

A l’ordre 1, on a:

- un doppler relativiste
- la déflexion de la lumière
- Changement de la vitesse de B pendant son temps de vol. B s’est déplacé avec une accélération en bonne approximation Newtonienne.

Numerical estimates of time delays in s over a year

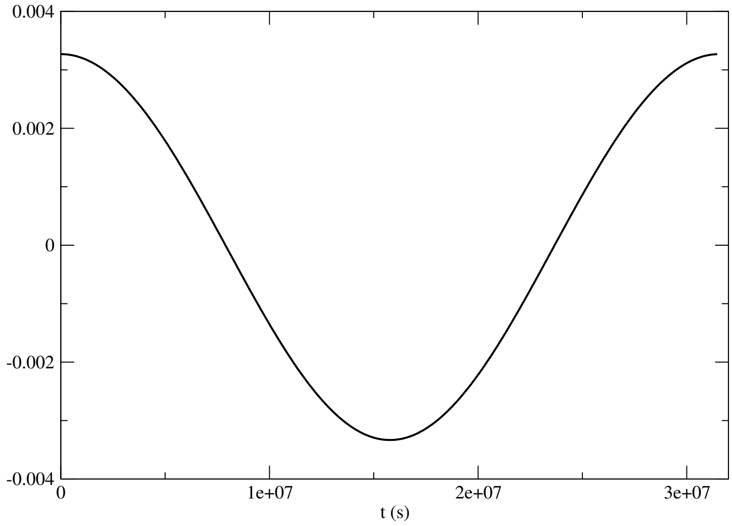
- t_{AB} **order 0 :** amplitude ~ 48 000 km/c



Numerical estimates of time delays in s over a year

- t_{AB} **order 0 :** « flexing » of triangle, amplitude ~ 48 000 km/c ;
- t_{AB} **order 1/2 :** amplitude ~ 960 km/c ;

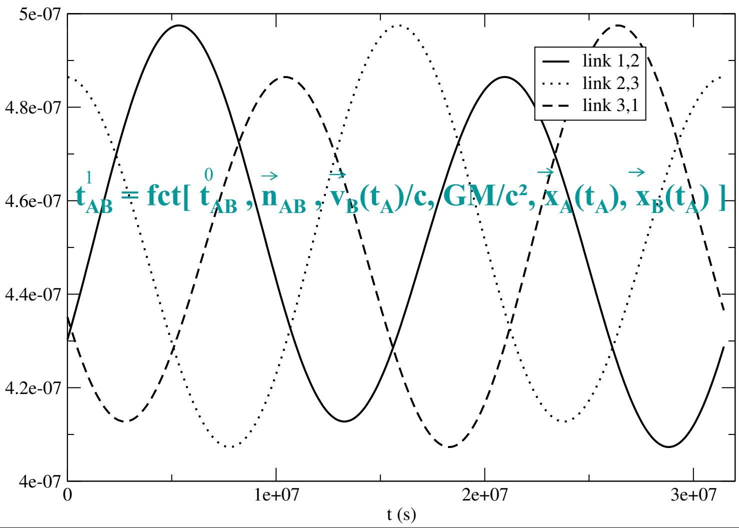
$t_{23}^{1/2} - t_{32}^{1/2} \ldots$ $t_{AB}^{1/2}$ is not symmetric **(Sagnac and aberration effects)**



Numerical estimates of time delays in s over a year

- t_{AB} **order 0 :** « flexing » of triangle, amplitude ~ 48 000 km/c ;
- t_{AB} **order 1/2 :** spacecraft Doppler, amplitude ~ 960 km/c ;
- t_{AB} **order 1 :** negligible, being less than 30 m/c.

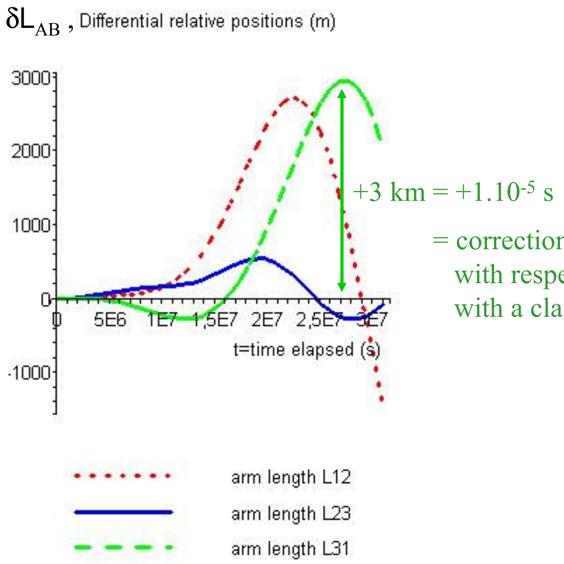
relativistic gravitational Einstein, Doppler, Shapiro effects



$$t^1_{AB} = \mathrm{fct}[\, t^0_{AB}, \vec{n}_{AB}, \vec{v}_B(t_A)/c, GM/c^2, \vec{x}_A(t_A), \vec{x}_B(t_A) \,]$$

Error in numerical time delays (over a year) if classical ephemerides are considered instead of GR ones

- t_{AB} **order 0 :** $\overset{0}{t}_{AB} = \frac{L_{AB}(t_A)}{c}$



= correction to zeroth order term with respect to time transfer with a classical orbitography

C4. LISA FREQUENCY SHIFT SOLUTION

[Chauvineau, Pireaux, Regimbau, Vinet, Phys. Rev. D 2005]

$$z_{AB} = \overset{1/2}{z}_{AB} + \overset{1}{z}_{AB} + \overset{3/2}{z}_{AB} + O(\varepsilon^2)$$

- **Order 1/2:** $\overset{1/2}{z} = -\overset{0}{\vec{n}} \cdot \frac{\vec{v}_{B0} - \vec{v}_{A0}}{c}$ → $\sim 7\ 10^{-8}$

  Kinematic terms (Doppler)

- **Order 1:**

$$\overset{1}{z} = \left(\overset{0}{\vec{n}} \cdot \frac{\vec{v}_{B0} - \vec{v}_{A0}}{c}\right)^2 - \frac{1}{2}\left(\frac{\vec{v}_{B0} - \vec{v}_{A0}}{c}\right)^2 + \frac{GM\, t\, \overset{0}{\vec{n}}_{AB} \cdot \overset{0}{\vec{x}}_{B0}}{c\, r_{B0}^3} + \frac{GM}{c^2}\left(\frac{1}{r_{B0}} - \frac{1}{r_{A0}}\right)$$

Kinematic terms: $\sim 2\ 10^{-15}$, $\sim 6\ 10^{-12}$

Velocity change during photon flight time: $\sim 2\ 10^{-10}$

Einstein effect: $\sim 2\ 10^{-10}$

Sum of the last two: $\sim 6\ 10^{-12}$

Total: $\sim 2\ 10^{-13}$

- **Order 3/2 :** $\overset{3/2}{z} \sim 2\ 10^{-14}$, as expected

A l'ordre ½, Doppler classique.

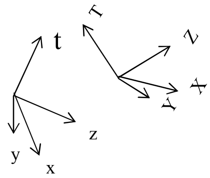
Dans le cas ou LISA serait un triangle équilatéral parfait, on aurait l'ordre ½ qui s'annulerait. Ici, on a une annulation presque parfaite, à cause du fait que les stations ont des mouvement circulaires autour du CM LISA et que leur vitesse est proportionnelle au rayon de l'orbite.

L'estimation précise montre que l'estimation naïve est réduite par un facteur L/R.

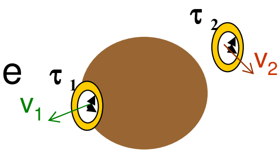
Part D- CONCLUSION

Need for *coherent and native* GR approach of modern space missions:

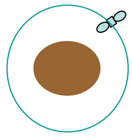
- spacetime coordinates and transformations



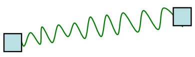
- clocks frequency and time



- orbitography



- laser links



Better than a classical + correction approach:

- universal approach to space missions: key = metric
- no effect or coupling omitted (with respect to chosen metric)
- …

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