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The observable light deflection angle

J.-M. Gérard and S. Pireaux

Institut de Physique Théorique

Université catholique de Louvain

B-1348 Louvain-la-Neuve, Belgium

To the memory of Jacques Demaret

Abstract

The physical deflection angle of a light ray propagating in a space-time supplied with an asymptotically flat metric has to be expressed in terms of the impact parameter.

1 Introduction

Light deflection by a gravitational source played a crucial role in the early establishment of General Relativity. Nowadays, gravitational optics is systematically used to search for elusive dark matter candidates (such as white dwarfs) through microlensing effects [1].

¿From a theoretical point of view, light deflection also provides us with a rather unique tool to test relativistic theories of gravity at various distance scales. In most of these theories, the exact value of the predicted deflection angle requires some numerical integration. Indeed, elliptic functions of the first kind are already necessary in the simple case of General Relativity. Therefore, approximate analytical expressions are often used in the literature, both in the weak and strong field limits. In this short note, we would like to point out that only expressions in terms of the physical impact parameter are meaningful. Although this statement is rather academic in the case of General Relativity, we illustrate how it might have quantitative consequences for extended theories of gravity.

2 The deflection angle ...

Let us consider a static and spherically symmetric space-time defined by the line element

$$ds^2 = A^2(r)c^2dt^2 - B^2(r)dr^2 - D^2(r)r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{1}$$

The null geodesics equations in the $\theta = \frac{\pi}{2}$ plane lead to the following equation for the photon trajectories :

$$\left(\frac{du}{d\varphi}\right)^2 + \frac{D^2}{B^2}u^2 = \frac{1}{b^2}\frac{D^4}{A^2B^2} \tag{2}$$

with

$$u \equiv \frac{1}{r}.$$

If space-time is asymptotically flat, b is simply the impact parameter which can be expressed in terms of the radial coordinate of the point (r_0, φ_0) of closest approach

$$b = \frac{D(r_0)}{A(r_0)} r_0. \tag{3}$$

Moreover, in such a space-time, the deflection angle $\hat{\alpha}$ is usually defined by

$$\begin{aligned}
\hat{\alpha}(r_0) &\equiv 2\int_0^{\varphi_0} d\varphi - \pi \\
&= 2\int_{r_0}^{\infty} \frac{B(r)}{D(r)} \Big\{ \Big(\frac{r}{r_0}\Big)^2 \frac{D^2(r)}{D^2(r_0)} \frac{A^2(r_0)}{A^2(r)} - 1 \Big\}^{-\frac{1}{2}} \frac{dr}{r} - \pi
\end{aligned} \tag{4}$$

and requires, in general, a numerical integration.

In the weak field approximation, the standard Eddington-Robertson parametrization is defined by

$$\begin{aligned}
A^2(\rho) &\simeq 1 + 2\alpha\left(\frac{V}{c^2}\right) + 2\beta\left(\frac{V}{c^2}\right)^2 + \frac{3}{2}\xi\left(\frac{V}{c^2}\right)^3 \\
B^2(\rho) &= D^2(\rho) \\
&\simeq 1 - 2\gamma\left(\frac{V}{c^2}\right) + \frac{3}{2}\delta\left(\frac{V}{c^2}\right)^2
\end{aligned} \tag{5}$$

with

$$V(\rho) \equiv -\frac{GM}{\rho},$$

the gravitational potential expressed in terms of the isotropic radial coordinate ρ. In this approximation, the deflection angle as a function of the closest approach distance reads

$$\hat{\alpha}_\omega(\rho_0) = \frac{4GM}{c^2\rho_0}\Big\{\Big(\frac{\alpha+\gamma}{2}\Big) + \Big[-\frac{(\alpha+\gamma)^2}{2} + \frac{(8\alpha^2 + 8\alpha\gamma - 4\beta + 3\delta)}{16}\pi\Big]\frac{GM}{c^2\rho_0}\Big\} + \mathcal{O}\Big(\frac{V^3}{c^6}\Big). \tag{6}$$

In particular, one easily proves the absence of light deflection in a space-time supplied with a conformally flat metric

$$g_{\mu\nu} = \left(1 + \frac{V}{c^2}\right)^2 \eta_{\mu\nu}. \tag{7}$$

3 ... in General Relativity ...

The Schwarzschild coordinates are more appropriate to study light deflection in the vicinity of a strong gravitational field. In the case of General Relativity, they correspond to the choice

$$\begin{aligned} A^2(r) &= B^{-2}(r) = 1 - \frac{2m}{r} \\ D^2(r) &= 1 \end{aligned} \tag{8}$$

with

$$m \equiv \frac{GM}{c^2}.$$

Assuming

$$r_0 \geq 3m \tag{9}$$

to ensure deflection, we obtain from Eqs. (4) and (8) the angle as an **exact** function of the closest distance of approach [2]

$$\hat{\alpha}(r_0) = 4\left(\frac{r_0}{q}\right)^{\frac{1}{2}} \left[F\left(\frac{\pi}{2}, k\right) - F(\sigma_0, k)\right] - \pi. \tag{10}$$

Here, the elliptic integral of the first kind

$$F(\sigma, k) \equiv \int_0^\sigma \frac{dx}{\{1 - k^2 \sin^2 x\}^{\frac{1}{2}}} \tag{11}$$

is characterized by the amplitude σ with

$$\sigma_0 = \arcsin\left\{\frac{q - r_0 + 2m}{q - r_0 + 6m}\right\}^{\frac{1}{2}} \tag{12}$$

and the modulus

$$k = \left\{\frac{q - r_0 + 6m}{2q}\right\}^{\frac{1}{2}} \tag{13}$$

where

$$q \equiv \left\{\left(1 - \frac{2m}{r_0}\right)\left(1 + \frac{6m}{r_0}\right)\right\}^{\frac{1}{2}} r_0.$$

In the weak field approximation

$$\epsilon \equiv \frac{m}{r_0} \ll 1 \tag{14}$$

and the modulus is small

$$k^2 = 4\epsilon(1-3\epsilon) + \mathcal{O}(\epsilon^3). \tag{15}$$

Expanding the elliptic integrals

$$\begin{aligned} F\left(\frac{\pi}{2}, k\right) &\simeq \frac{\pi}{2}\left(1+\epsilon-\frac{3}{4}\epsilon^2\right) \\ F\left(\sigma_0, k\right) &\simeq \frac{\pi}{4}\left(1+\epsilon-\frac{3}{4}\epsilon^2\right) - \epsilon \end{aligned} \tag{16}$$

we obtain

$$\hat{\alpha}_\omega(r_0) \simeq \frac{4GM}{c^2 r_0}\left\{1+\left[-1+\frac{15}{16}\pi\right]\frac{GM}{c^2 r_0}\right\}. \tag{17}$$

in the Schwarzschild coordinates. On the other hand, Eq. (6) implies

$$\hat{\alpha}_\omega(\rho_0) \simeq \frac{4GM}{c^2 \rho_0}\left\{1+\left[-2+\frac{15}{16}\pi\right]\frac{GM}{c^2 \rho_0}\right\} \tag{18}$$

in the isotropic coordinates, since the Eddington-Robertson parameters defined in Eq. (5) are conventionally normalized to unity for General Relativity.

The apparent second order discrepancy between Eqs. (17) and (18) can be understood in the following way. The deflection angle being by definition observable, it has to be fully expressed in terms of measurable, i.e. coordinate-independent, quantities. Here, the closest distance of approach (r_0 in Schwarzschild coordinates, ρ_0 in isotropic coordinates) is obviously not such a measurable quantity. Its corresponding substitution by the impact parameter b according to Eq. (3)

$$\begin{aligned} b &\simeq r_0 + m \\ &\simeq \rho_0 + 2m \end{aligned} \tag{19}$$

reconciliates our results of calculations performed with two physically equivalent forms of the metric. In General Relativity, the observable deflection angle is correctly given by

$$\hat{\alpha}_\omega(b) = \frac{4GM}{c^2 b}\left\{1 + \left[0 + \frac{15\pi}{16}\right]\frac{GM}{c^2 b}\right\} + \mathcal{O}\left(\frac{m^3}{b^3}\right). \tag{20}$$

This is the analog of what has been noted [3] for the radar-echo experiment where the observable transit time has to be expressed in terms of the measurable orbital parameters (periods and eccentricities).

In the strong field approximation

$$\epsilon \equiv 1 - \frac{3m}{r_0} \ll 1 \tag{21}$$

and the modulus is close to one

$$k^2 = 1 - \frac{4}{3}\epsilon + \mathcal{O}(\epsilon^2). \tag{22}$$

Expanding again the elliptic integrals

$$\begin{aligned} F\left(\frac{\pi}{2}, k\right) &\simeq \ln\frac{4}{\sqrt{1-k^2}} \\ F\left(\arcsin\frac{1}{\sqrt{3}}, 1\right) &\simeq 0.65848, \end{aligned} \tag{23}$$

we obtain a simple analytical expression for the deflection angle of a light ray grazing a black hole[2] :

$$\hat{\alpha}_s(b) \simeq \ln\left(\frac{3.482m}{b - 3\sqrt{3}m}\right). \tag{24}$$

The Eqs. (20) and (24) provide us with useful (weak and strong field) approximations to study lensing effects in terms of the observable deflection angle $\hat{\alpha}(b)$. They readily replace the unpractical exact expression obtained from Eq. (10) after substitution of r_0 by b according to Eqs. (3) and (8). However, the second order contribution to the deflection angle in Eq. (20) only improves the weak field approximation in a narrow domain of the impact parameter (see Fig. 1). In that sense, our comment on the ambiguous coordinate dependence of $\hat{\alpha}(r_0)$ defined in Eq. (4) is rather academic as far as General Relativity is concerned.

4 ... and beyond

Let us now consider the coupled Einstein-improved scalar action

$$S = -\frac{1}{2\kappa}\int d^4x\sqrt{-g}R + \frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu} + \frac{1}{6}R\Phi^2\right] \tag{25}$$

with

$$\kappa \equiv \frac{8\pi G}{c^4}.$$

The non-minimal coupling of the massless scalar Φ to the curvature invariant ensures the vanishing of the stress tensor's trace, as required by conformal symmetry [4].

It is however well-known [5] that an appropriate rescaling of the metric

$$\tilde{g}_{\mu\nu} = \left(1 - \frac{\kappa}{6}\Phi^2\right) g_{\mu\nu} \tag{26}$$

together with a redefinition of the scalar field

$$\Phi = \left(\frac{6}{\kappa}\right)^{\frac{1}{2}} \tanh\left[\left(\frac{\kappa}{6}\right)^{\frac{1}{2}}\widetilde{\Phi}\right] \tag{27}$$

recast the theory into the minimal form

$$\widetilde{S} = -\frac{1}{2\kappa}\int d^4x\sqrt{-\tilde{g}}\widetilde{R} + \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\widetilde{\Phi}_{,\mu}\widetilde{\Phi}_{,\nu}. \tag{28}$$

The most general static, spherically symmetric and asymptotically flat exact solution to the corresponding equations of motion

$$\begin{aligned} \widetilde{R}_{\mu\nu} &= \kappa\ \widetilde{\Phi}_{,\mu}\widetilde{\Phi}_{,\nu} \\ \Box\ \widetilde{\Phi} &= 0 \end{aligned} \tag{29}$$

is given by [6]

$$\begin{aligned} A^2(r) &= B^{-2}(r) = \left(1 - \frac{2m}{\lambda r}\right)^{\lambda} \\ D^2(r) &= \left(1 - \frac{2m}{\lambda r}\right)^{1-\lambda} \end{aligned} \tag{30}$$

for the metric, and

$$\widetilde{\Phi} = \left(\frac{1-\lambda^2}{2\kappa}\right)^{\frac{1}{2}} \ln\left(1 - \frac{2m}{\lambda r}\right) \tag{31}$$

for the scalar field. If we assume a positive Newton constant G (i.e. $\kappa > 0$), then

$$0 < \lambda \leq 1 \tag{32}$$

and we recover the standard Schwarzschild solution of General Relativity when λ goes to one.

Light deflection requires now

$$r_0 > \left(2 + \frac{1}{\lambda}\right) m. \tag{33}$$

In the weak field approximation, Eqs. (4) and (30) imply

$$\hat{\alpha}_\omega(r_0) \simeq \frac{4GM}{c^2 r_0}\left\{1 + \left[-2 + \frac{1}{\lambda} + \left(1 - \frac{1}{16\lambda^2}\right)\pi\right]\frac{GM}{c^2 r_0}\right\} \tag{34}$$

On the other hand, the Eddington-Robertson parameters of the minimal Einstein-massless scalar theory read

$$\begin{aligned} \alpha &= 1 \\ \beta &= 1 \\ \gamma &= 1 \\ \delta &= \frac{4}{3}\left(1 - \frac{1}{4\lambda^2}\right) \end{aligned} \tag{35}$$

such that

$$\hat{\alpha}_\omega(\rho_0) \simeq \frac{4GM}{c^2 \rho_0}\left\{1 + \left[-2 + \left(1 - \frac{1}{16\lambda^2}\right)\pi\right]\frac{GM}{c^2 \rho_0}\right\}. \tag{36}$$

Since a deviation from General Relativity only arises at the second order, the use ofthe physical impact parameter b according to Eq. (3)

$$\begin{aligned} b &\simeq r_0 + \left(2 - \frac{1}{\lambda}\right) m \\ &\simeq \rho_0 + 2m \end{aligned} \tag{37}$$

is now crucial to obtain the observable deflection angle

$$\hat{\alpha}_\omega(b) = \frac{4GM}{c^2 b}\left\{1 + \left(1 - \frac{1}{16\lambda^2}\right)\pi\frac{GM}{c^2 b}\right\} + \mathcal{O}\left(\frac{m^3}{b^3}\right). \tag{38}$$

The same result is obtained by working directly in the improved action basis characterized by

$$\begin{aligned} \alpha &= 1 \\ \beta &= \frac{5}{6} + \frac{1}{6\lambda^2} \\ \gamma &= 1 \\ \delta &= \frac{10}{9} - \frac{1}{9\lambda^2}. \end{aligned} \tag{39}$$

The authors of ref. [7] only gave the coordinate-dependent Eq. (4) to analyse the quantitative modifications of lensing characteristics in the presence of the massless scalar field $\widetilde{\Phi}$. We have just argued that Eqs. (4) and (3) should always be carefully handled to get rid of the closest distance of approach dependence. The deflection angle $\hat{\alpha}$ is then correctly expressed in terms of the physical impact parameter and does not depend on the arbitrary choice of coordinate system.

5 Conclusion

Any observable has to be expressed in terms of measurable quantities which are coordinate-independent. The use of the impact parameter is therefore mandatory for the deflection angle of a light ray propagating in a space-time supplied with an asymptotically flat metric. To our knowledge, this point has been overlooked in the literature.

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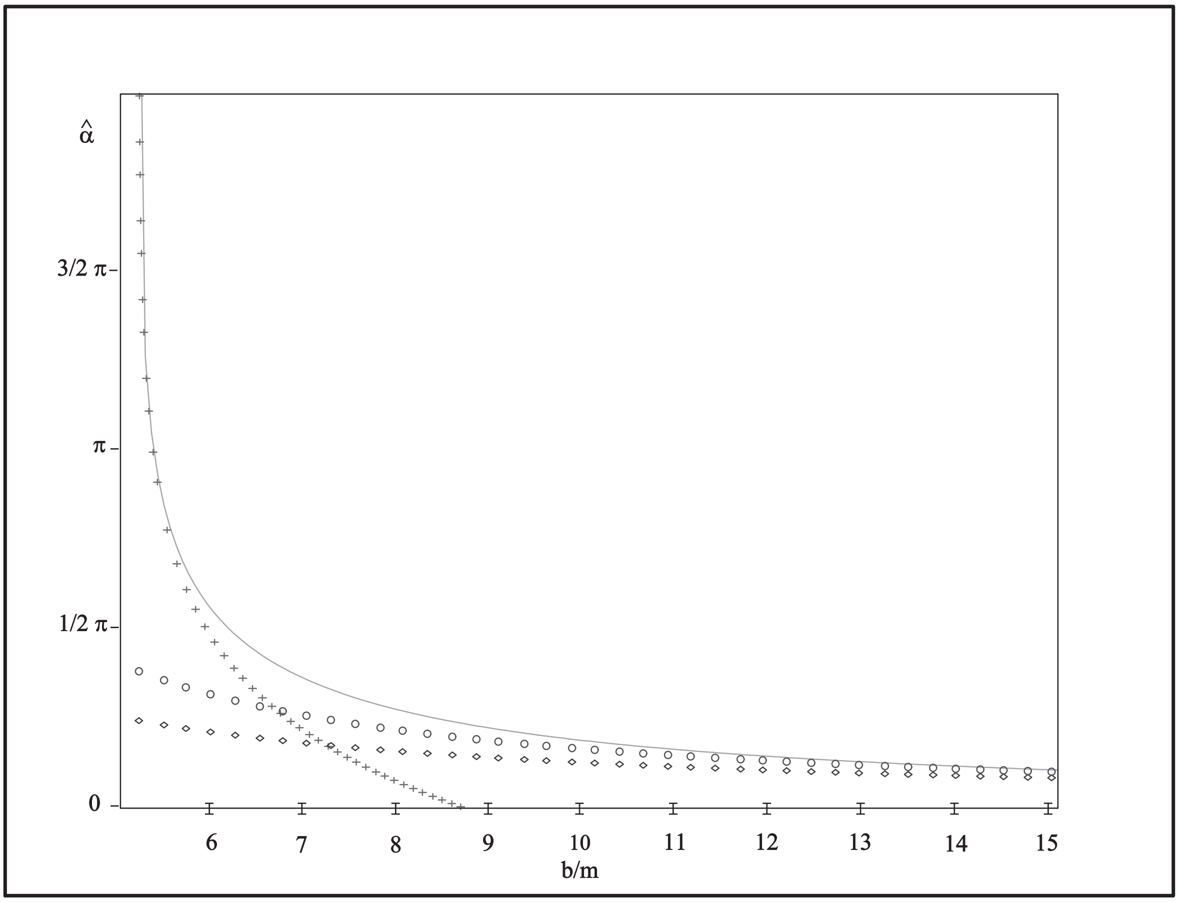

Figure 1: The light deflection angle $\hat{\alpha}$ predicted by General Relativity, as a function of the impact parameter b. The smooth curve corresponds to the exact expression based on Eq. (10) after substitution of the closest distance of approach r_0 by b according to Eqs. (3) and (8). Crosses stand for the strong field approximation given in Eq. (24). Diamonds and circles represent respectively the first and second order approximations in the weak field limit defined by Eq. (20).