

(SC)RMI:  
A (S)emi-(C)lassical (R)elativistic (M)otion  
(I)ntegrator, to model the orbits of space  
probes around the Earth and other planets

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**Abstract**

Today, the motion of spacecrafts is still described according to the classical Newtonian equations plus the so-called relativistic corrections, computed with the required precision using the Post-(Post-) Newtonian formalism. The current approach, with the increase of tracking precision (Ka-Band Doppler, interplanetary lasers) and clock stabilities (atomic fountains) is reaching its limits in terms of complexity, and is furthermore error prone. In the appropriate framework of General Relativity, we study a method to numerically integrate the native relativistic equations of motion for a weak gravitational field, also taking into account small non-gravitational forces. The latter are treated as perturbations, in the sense that we assume that both the local structure of space-time is not modified by these forces, and that the unperturbed satellite motion follows the geodesics of the local space-time. The use of a symplectic integrator to compute the unperturbed geodesic motion insures the constancy of the norm of the proper velocity quadrivector. We further show how this general relativistic framework relates to the classical one.

Keywords: orbitography, non-gravitational forces, General Relativity

# 1 The classical approach: *GINS*

In today’s planetary orbitography softwares, as in *GINS* (Géodésie par Intégrations Numériques Simultanées, developed by CNES<sup>1</sup> and GRGS<sup>2</sup>), the motion of spacecrafts is still described according to the classical Newtonian equations plus the so-called “relativistic corrections”, computed with the required precision using the Post-(Post-) Newtonian formalism.

Hence, it is the 3-dimensional acceleration vector ( $i = 1, 2, 3$ ), which is numerically integrated with respect to coordinate time  $T$ ,

$$\frac{d^2 X^i}{dT^2} = -\frac{\partial W}{\partial X^i} - K^i + \textit{general relativistic corrections.} \quad (1)$$

The gravitational potential  $W$  includes not only the central planetary potential model but also the Earth-tide potential (due to the Sun and Moon, corrected for Love number according to the frequencies, ellipticity and polar tides), ocean-tide potential and Newtonian-perturbation potentials from other solar system bodies. The atmospheric drag, the radiation pressure (solar radiation, Earth albedo, thermal emission) are the non-gravitational perturbations considered,  $K^i$ . The orbitography software *GINS* also includes, as relativistic corrections, the Schwarzschild, geodesic and Lense-Thirring precessions [1]. Figure 1 summarizes the *GINS* approach. It is a generic one which can be applied to orbitography around any central planet.

When analysing satellite data for geophysical reduction, General Relativity not only plays a role in computing the precise satellite orbit through equation (1). Indeed, general relativistic corrections are also applied on measurements, because electromagnetic signals travel in curved space-time, and are thus deflected and delayed. Also, the planetary potential model is described in the planetary crust frame, rotating with the central planet; while the satellite motion is described in the planetary quasi-inertial frame, non rotating with the planet. Both frames are linked through the planetary rotation model. This model must include the Earth relativistic geodesic precession, [2], [3], i.e.  $\sim 2$  arcseconds/century mostly due to the Sun static gravitational field, a general relativistic correction stemming from the fact that the Earth frame is not an inertial frame but moves in the gravitational field generated by other solar system bodies.

Furthermore, any reference frame transformation (geocentric to barycentric and vice-versa, needed to get positions and velocities of solar system bodies from ephemerides) is a 4-dimensional space-time transformation in General Relativity. This means adding more relativistic corrections to the corresponding Newtonian 3-space transformations and abolishing the status of an absolute time. Time is intricately related with space, and time is relative to a reference system.

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## 2 Motivations

### 2.1 Relevance of relativistic effects

To illustrate the relevance of relativistic corrections on a classical orbit, let us consider LAGEOS as an example of a high-orbit satellite, at an altitude of about 6000 km, designed for geodynamic studies. As an example of a low satellite, we take CHAMP, dedicated to the precise determination of the Earth gravitational field, flying at a mean altitude of 450 km. Table 2.1 illustrates our point. Relativistic corrections are crucial for a high orbit, because they are of the same order of magnitude as the most important non-gravitational effect, namely radiation pressure. As a low-orbit satellite is concerned, general relativistic effects are comparable in order of magnitude to thermal emission or radiation pressure. But, of course, at low altitudes, the atmospheric drag is much more important.

Figures 2, 3 and 4 detail the relativistic corrections already included in the GINS software, and provide orders of magnitude of the corresponding accelerations induced.

The Schwarzschild precession is the most important one. It is associated with the Earth mass monopole and mass multipoles, which deforms the space-time and hence affects the satellite motion. It leads to a perigee advance of the orbit of the satellite around the Earth, about 3 arcsec/year for LAGEOS 1.

Like the Schwarzschild precession, the geodesic precession is associated with the Earth mass. It is due to the fact that the Earth “warps” the space-time around itself. Gravity Probe B, a NASA mission, should be able to measure this effect, which is about 6.614 marcsec/year for gravity Probe B (on a polar orbit at 640 km), with a relative precision of  $2 \cdot 10^{-5}$ .

The Lense-Thirring general relativistic effect is associated with the Earth spin dipole. As the Earth rotates, it “drags” the space-time, like a spinning ball in molasses. The Lense-Thirring effect leads to an additional perigee precession of the orbit of the satellite, plus a secular drift of the ascending node. For LAGEOS 1, it is about 3 arcsec/century. For Gravity Probe B on a polar orbit, the Lense-Thirring and geodesic precessions are orthogonal. This order of magnitude is comparable to the Schwarzschild effect due to the degree 12 mass multipole moment of the Earth. Gravity Probe B, for which the Lense-Thirring precession amounts to about 40.9 marcsec/year, should be able to measure this general relativistic precession with a relative precision of  $3 \cdot 10^{-3}$ .

### 2.2 Precise geodesy for precise geophysics

The present study is motivated by the fact that precise geophysics requires precise geodesy.

From the above paragraphs, we see that errors in relativistic corrections or relativistic space-time transformations between reference frames lead to a mis-modelling in the planetary potential and rotation model deduced from precise orbitography. There is a real risk of polluting very weak geophysical signatures, like the polar motion of Mars ( $\sim 1$  m in amplitude at the planet surface), or the nutations of its conjectured liquid core ( $\sim$  a few cm over an amplitude of  $\sim 10$  m), by unwanted relativistic effects that are at the same period (typically one planetary year, or 687 days for Mars), and, worse, that can be cumulative (ranging error up to or larger than 10 m coming from relativity over one spacecraft orbit around Mars,  $\sim 150$  minutes). Moreover, with the classical method, one correction can sometimes be counted twice (for example, the reference frequency provided by GPS satellites is already corrected for the main relativistic effect), if not forgotten.

### 2.3 Disadvantages of the classical method

With the increase of tracking precision (32 GHz Ka/Ka-Band Doppler radio tracking at the level of  $1 \mu\text{m/s}$  with respect to a relative motion Earth/spacecraft of 10 km/s, i.e. with a relative accuracy of  $10^{-10}$ ), active interplanetary laser tracking (at the level of 10 cm with respect to a distance of  $10^8$  km, i.e. with a relative accuracy of  $10^{-12}$ ) and clock stabilities (Allan deviation of  $\sim 4 \cdot 10^{-14} \tau^{-1/2}$  for atomic fountains), the classical method is today reaching its limits in terms of complexity. A (complete) review of all the corrections is needed in case of any change in conventions (metric adopted), or if further precision is gained in measurements.

This is why, we suggest to use general relativistic mechanics from the beginning, instead of Newtonian mechanics plus an increasing number of corrections.

## 3 The (Semi-Classical) relativistic approach: (SC)RMI

### 3.1 Relativistic equations

In general relativistic mechanics, the relativistic equation of motion, when non-gravitational accelerations (encoded in a 4-vector [5]  $K_\beta$ ) are present, is

$$\frac{dU^\alpha}{d\tau} = -\Gamma_{\beta\gamma}^\alpha U^\beta U^\gamma + K_\beta \left( G^{\alpha\beta} - \frac{U^\alpha U^\beta}{c} \right) \quad (2)$$

$$\text{with } U^\alpha \equiv \frac{dX^\alpha}{d\tau}, \quad U^\alpha U_\alpha = c^2 \quad (3)$$

where  $X^{\alpha=0,1,2,3} \equiv (c \cdot T, X^i)$  are the space-time coordinates;  $\Gamma_{\beta\gamma}^\alpha$  are the Christoffel symbols;  $G^{\alpha\beta}$ , space-time metric;  $c$  is the speed of light; and  $\tau$  is the proper time. When  $K_\beta = 0$ , equation (2) reduces to the geodesic equation of the local space-time.

### 3.2 (SC)RMI method

We propose to numerically integrate the native relativistic equations of motion for a weak gravitational field in the appropriate framework of General Relativity, taking into account not only gravitational forces, but also non-gravitational ones. In other words, it is those 4-dimensional equations (i.e.  $\frac{d^2 X^\alpha}{d\tau^2}$ ) which are directly numerically integrated. This new approach is called (SC)RMI ((Semi-Classical) Relativistic Motion Integrator). RMI was first suggested by X. Moisson and S. Loyer during their PH.D. Thesis at the Observatoire de Paris [4]. Figure 5 summarizes the (SC)RMI approach.

For the appropriate metric at the required order (power series in  $1/c^2$ ), the general relativistic equations (2) contain all the gravitational effects at the corresponding order. They are coded in the Christoffel symbols that are functions of the derivative of the metric. The metric  $G^{\alpha\beta}$  itself encodes the geometry of space-time that is shaped (deformed) by the presence of solar system bodies. The above equations (2) computed for the Geocentric Coordinate Reference System (GCRS) metric [2], [3] will take into account gravitational multipole moment contributions from the central planetary gravitational potential, perturbations due to solar system bodies, the Schwarzschild, geodesic and Lense-Thirring precessions.

Non-gravitational forces can be treated as perturbations, in the sense that they do not modify the local structure of space-time (the metric). Moreover,  $K_\beta$  being small, one can safely replace  $G^{\alpha\beta}$  by its Minkowskian counterpart,  $\eta^{\alpha\beta}$ , in the second term of the right-hand-side of equation (2), hence the terminology ‘‘Semi-Classical’’ in SCRMI. This leads to

$$\frac{dU^\alpha}{d\tau} = -\Gamma_{\beta\gamma}^\alpha U^\beta U^\gamma + K_\beta \left( \eta^{\alpha\beta} - \frac{U^\alpha U^\beta}{c} \right) \quad (4)$$

In the classical limit, equations (2, 4) reduce to the Newtonian gravitational 3-acceleration equation (1) (without relativistic corrections) where  $W$  is a generalized gravitational potential given in the GCRS metric.

### 3.3 A symplectic integrator

Expression (2) consists in fact of four equations, to compare with the three equations (1) to be integrated in Newtonian mechanics. However, a first integral exists, i.e. equation (3), since the norm (with respect to the metric  $G^{\alpha\beta}$ ) of the quadri-proper-velocity,  $U^\alpha$ , is conserved along the trajectory. This point stresses the importance of a symplectic integrator naturally preserving this quantity.

### 3.4 (SC)RMI validation

(SC)RMI, based on the relativistic equations (4), is validated by comparison with template orbits from GINS in which a particular relativistic effect has been

switched on. For example, using, in (SC)RMI, the Schwarzschild metric (based on a single central spherical static body) allows to validate the Schwarzschild precession. If a multipolar geopotential model is added to the Schwarzschild mass monopole, one can further validate the harmonic contributions of the potential. Using the Kerr metric (based on a single central spherical rotating gravitational body) allows to validate an additional relativistic effect: the Lense-Thirring precession. Finally, using the GCRS metric taking into account the Solar and planetary gravitational fields will include additional effects like the geodesic precession and the perturbations from the corresponding planets. Once completed, (SC)RMI will go beyond GINS, including the IAU2000/IERS 2003 new standards regarding metric, space-time transformations and Earth rotation models. Moreover, separate modules allow easy updates for metric, Earth potential... IAU recommendations, keeping the integrator body unchanged. It is also important to stress that the (SC)RMI approach natively contains not only all the relativistic effects at the corresponding order of the metric, but also all the couplings between these effects at the corresponding order.

## 4 The principle of accelerometers

Last we show how to update the classical equation for accelerometers; in other words, how to measure  $K_\beta$ , or how to introduce a non-gravitational force model in the relativistic framework.

Let the satellite center of mass (CM) be located at  $X^\mu$ ; while a test-mass is at  $X^\mu + \delta X^\mu$ , in a cavity inside the satellite, hence shielded from non-gravitational forces. The test-mass motion is described by geodesic equations ((2) with  $K_\beta=0$ ) while that of the satellite is described by (2). Evaluating the difference between those two equations at first order in  $\delta X^\mu$  gives a general relativistic equation for accelerometer measurements:

$$\begin{aligned} \frac{d^2 \delta X^\alpha}{d\tau^2} = & +K_\beta^{(CM)} \left( G^{\alpha\beta} - \frac{dX^\alpha}{d\tau} \frac{dX^\beta}{d\tau} \right) \\ & - \frac{\partial \Gamma_{\beta\gamma}^\alpha}{\partial X^\mu} \delta X^\mu \frac{dX^\beta}{d\tau} \frac{dX^\gamma}{d\tau} \\ & - 2\Gamma_{\beta\gamma}^\alpha \frac{dX^\beta}{d\tau} \frac{d\delta X^\gamma}{d\tau} \end{aligned} \quad (5)$$

Equation (5) reduces to geodesic deviation if  $K_\beta^{(CM)}=0$ .

In the classical limit, one recovers the classical accelerometer equation,

$$\frac{d^2 \delta X^i}{dT^2} = -K_i^{(CM)} - \sum_j \frac{\partial^2 W}{\partial X^i \partial X^j} \delta X^j \quad (6)$$

where  $\frac{d^2 \delta X^i}{dT^2}$  is the 3-dimensional relative acceleration of the satellite with respect to the test mass; the third term of equation (5) vanishes completely.

## 5 Conclusions

We have outlined here a new paradigm for orbitography software that is, in our opinion, a key for the future. The concepts differs radically from those in use today. To implement this idea requires rewriting the core parts of existing programs; they cannot be simply “upgraded”. This is obviously the main difficulty to overcome. The (SC)RMI prototype software can be validated using a classical orbitography software thanks to the progressive method described in Section 3.4. Some of step of this method have already been implemented. In reference “Non-gravitational forces and the relativistic equations of motion” [6] in preparation, the authors provide further details on symplectic integrators (mentioned in Section 3.3 of the present article) and on the non-gravitational force term (to be measured from accelerometers as described in Section 4, or provided by classical models).

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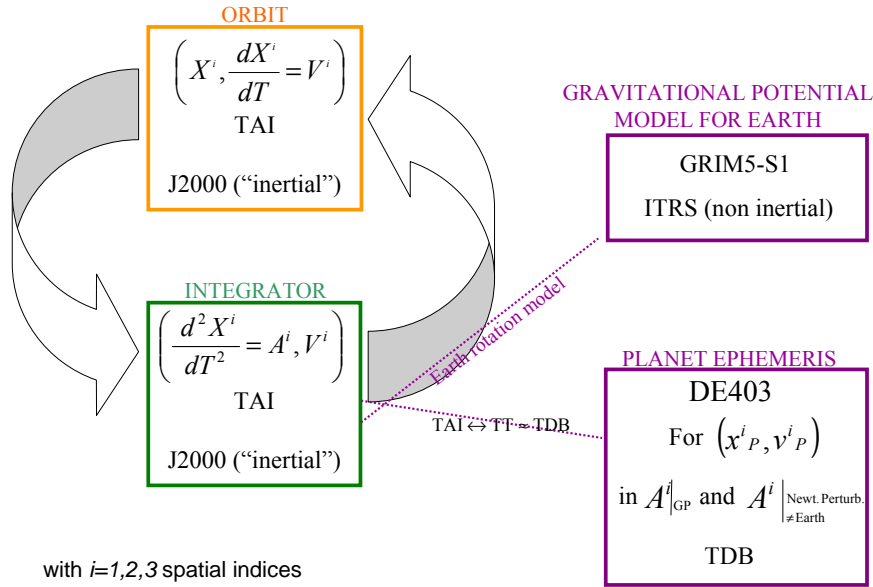


Figure 1: GINS.

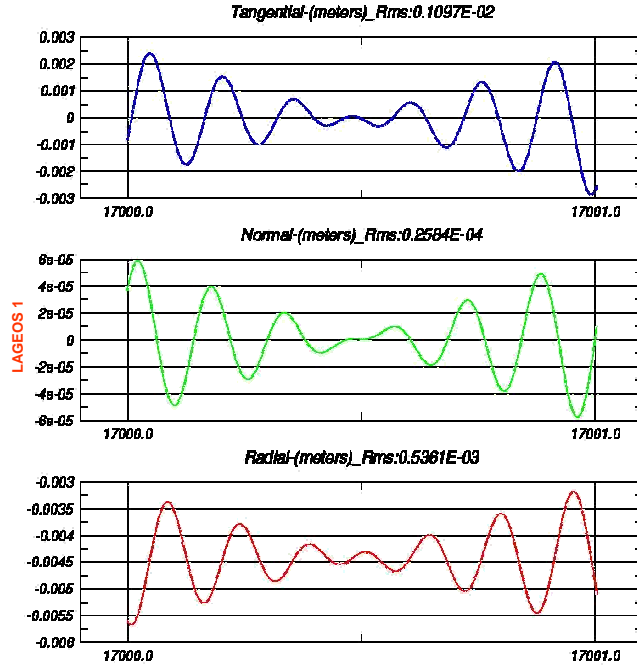
This diagram summarizes the GINS orbitography software approach. The orbit of the satellite is given in terms of 3-dimensional position and velocity vectors,  $X^i$  and  $V^i = dX^i/dT$  respectively, in the J2000 quasi-inertial Earth frame, with respect to International Atomic Time (TAI). It is the 3-dimensional acceleration,  $A^i = d^2 X^i/dT^2$ , and velocity vectors which are integrated with respect to this time. The central gravitational potential can be either the Earth (E) potential given for example by the GRIM5-S1 model provided in the International Terrestrial Reference System (ITRS) or any central body planetary potential, completed by the potential of other celestial bodies. Velocities and positions of planets (P), that is  $x^i_P$  and  $v^i_P$ , provided in terms of TDB (“Temps Dynamique Barycentrique”) in the barycentric reference frame by planetary ephemeris (such as DE403) are needed to compute the Newtonian contribution from planets as well as relativistic accelerations (Geodesic Precession (GP)). Additional space (Earth or central body rotation model) and time (TAI  $\leftrightarrow$  TDB, with TDB assumed in GINS to be equal to the Terrestrial Time (TT)) are needed to transform between the different reference frames.

Cause (m/s <sup>2</sup> )	LAGEOS 1	CHAMP
<b>G:</b> Earth monopole	2.8	8.6
<b>G:</b> Low order geopotential harmonics (eg. l=2,m=2)	6.0 10 <sup>-6</sup>	6.4 10 <sup>-5</sup>
<b>G:</b> High order geopotential harmonics (eg. l=18,m=18)	6.9 10 <sup>-12</sup>	9.4 10 <sup>-7</sup>
<b>G:</b> Moon	2.1 10 <sup>-6</sup>	7.9 10 <sup>-7</sup>
<b>G:</b> Sun	9.6 10 <sup>-7</sup>	2.7 10 <sup>-7</sup>
<b>G:</b> Other planets (eg. Venus)	1.3 10 <sup>-10</sup>	9.8 10 <sup>-13</sup>
<b>G:</b> Indirect oblation (Moon-Earth)	1.4 10 <sup>-11</sup>	1.4 10 <sup>-11</sup>
<b>G:</b> General relativistic corrections (total)	9.5 10 <sup>-10</sup>	1.7 10 <sup>-8</sup>
<b>NG:</b> Atmospheric drag	3 10 <sup>-12</sup>	3.5 10 <sup>-7</sup>
<b>NG:</b> Solar radiation pressure	3.2 10 <sup>-9</sup>	3.2 10 <sup>-8</sup>
<b>NG:</b> Earth albedo pressure	3.4 10 <sup>-10</sup>	3.3 10 <sup>-9</sup>
<b>NG:</b> Thermal emission	1.9 10 <sup>-12</sup>	8.3 10 <sup>-9</sup>

Table 1: Gravitational and non-gravitational satellite accelerations.

Orders of magnitude of the gravitational (G) or non-gravitational (NG) accelerations in m/s<sup>2</sup> experienced by a high- or a low-orbit satellite (LAGEOS 1 or CHAMP respectively). Discrepancies between accelerations on LAGEOS 1 or on CHAMP due to planets (especially Venus) come from the fact that the value was estimated at a different Julian Days, when the position of the corresponding planet in the solar system was different. The Laser GEODYNAMICS Satellite 1 was designed to calculate station positions (with a precision of about 1-3 cm), monitor tectonic-plate motion, measure the Earth gravitational field and Earth rotation. It is a passive (no onboard sensors/electronic, no attitude control) spherical satellite with laser reflectors, on a 5858 x 5958 km,  $i = 52.6^\circ$ , orbit around Earth. The mission was launched in 1976 by the USA for an minimum lifetime of 50 years. The CHALLENGING Minisatellite Payload is a German mission for precise gravity and magnetic field modeling including space and time variations. It holds laser reflectors, a GPS receiver, a drift meter, a magnetometer, a star sensor and accelerometers. It was launched in 2000 for a five-year mission on a near polar Earth orbit at an initial altitude of 454 km.

Orbit comparison GINS: Schwarzschild effect



$$\vec{A}_{\text{Schw}} = \frac{GM_E}{c^2 |\vec{X}|^3} \left[ \left( \frac{4GM_E}{|\vec{X}|} - |\vec{V}|^2 \right) \vec{X} + 4(\vec{V} \cdot \vec{X}) \vec{V} \right]$$

Figure 2: Schwarzschild precession.

The Schwarzschild general relativistic precession,  $\vec{A}_{\text{Schw}} \equiv d^2 \vec{X} / dT^2|_{\text{Schw}}$ , due to the Earth mass monopole,  $GM_E$ , is function of the satellite geocentric velocity,  $\vec{V}$ , and position,  $\vec{X}$  with respect to the Geocentric Coordinate Reference System (GCRS). This effect is illustrated here with a one day data arc of satellite LAGEOS 1 by comparing two orbits generated by GINS, with or without that effect included. Tangential, normal and radial effects (rms) are provided in m.

Orbit comparison GINS: Geodetic effect

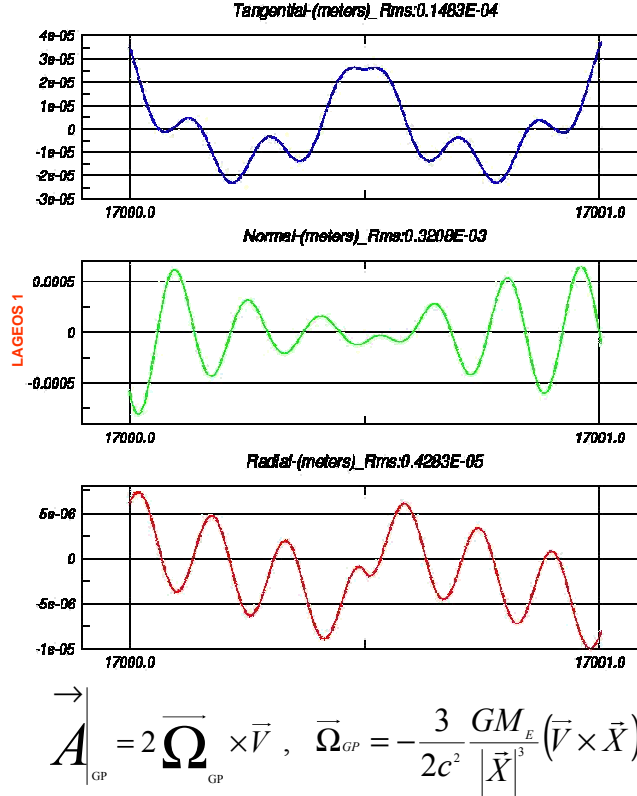
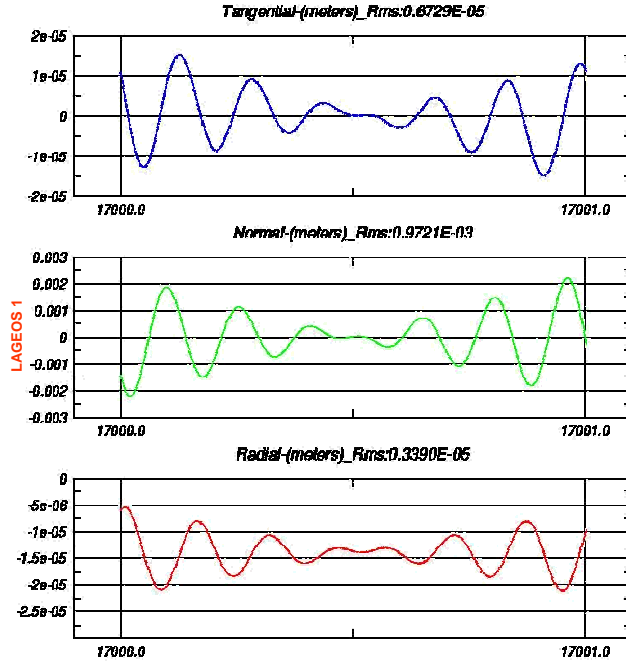


Figure 3: Geodesic precession.

The geodesic general relativistic precession,

$\vec{A}_{GP} \equiv d^2 \vec{X} / dT^2|_{GP}$ , due to the Earth mass monopole,  $GM_E$ , is function of the satellite geocentric velocity,  $\vec{V}$ , and position,  $\vec{X}$  with respect to the Geocentric Coordinate Reference System (GCRS). This effect is illustrated here with a one day data arc of satellite LAGEOS 1 by comparing two orbits generated by GINS, with or without that effect included. Tangential, normal and radial effects (rms) are provided in m.

Orbit comparison GINS: Lense-Thirring effect

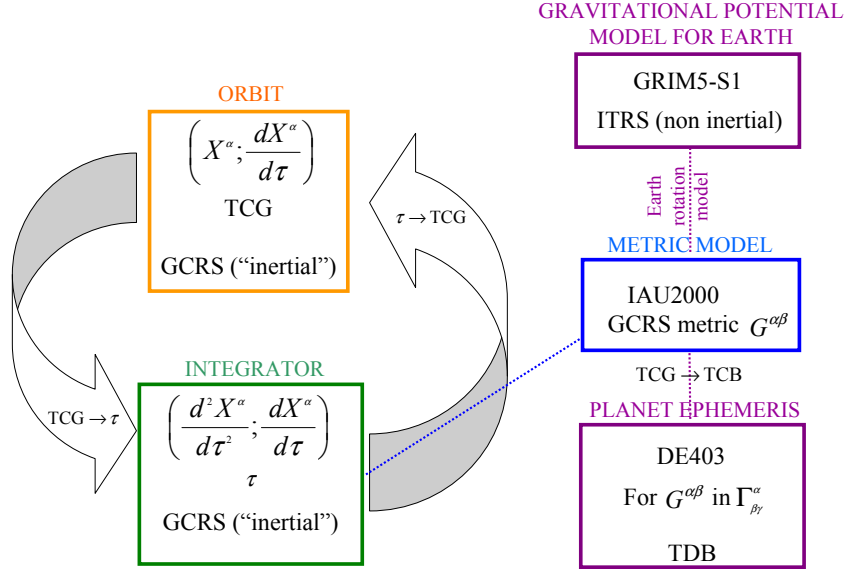


$$\vec{A}\Big|_{\text{LTP}} = 2 \vec{\Omega}_{\text{LTP}} \times \vec{V}, \quad \vec{\Omega}_{\text{LTP}} = -\frac{G}{c^2 |\vec{X}|^3} \left( + \vec{S}_E - \frac{3(\vec{S}_E \cdot \vec{X}) \vec{X}}{|\vec{X}|^2} \right)$$

Figure 4: Lense-Thirring precession.

The Lense-Thirring general relativistic precession,

$\vec{A}\Big|_{\text{LTP}} \equiv d^2 \vec{X} / dT^2\Big|_{\text{LTP}}$ , due to the Earth spin dipole,  $\vec{S}_E$ , is function of the satellite geocentric velocity,  $\vec{V}$ , and position,  $\vec{X}$  with respect to the Geocentric Coordinate Reference System (GCRS). This effect is illustrated here with a one day data arc of satellite LAGEOS 1 by comparing two orbits generated by GINS, with or without that effect included. Tangential, normal and radial effects (rms) are provided in m.



with  $\alpha=0,1,2,3$  space-time indices

Figure 5: (SC)RMI.

This diagram summarizes the (SC)RMI approach. The orbit is given in terms of 4-dimensional position and velocity proper-vectors of the satellite,  $X^\alpha$  and  $dX^\alpha/d\tau$  respectively (proper meaning with respect to the satellite proper time). These quantities are referred to the J2000 Geocentric Coordinate Reference System (GCRS) Earth frame and its corresponding TCG time (“Temps de Coordonnée Géocentrique”). It is the 4-dimensional proper-acceleration,  $d^2 X^\alpha/d\tau^2$ , given by the relativistic equations of motion and the 4-dimensional proper-velocity vectors which are integrated with respect to proper time  $\tau$ , instead of coordinate time TCG. To compute the relativistic equations, one needs to compute the GCRS metric,  $G_{\alpha\beta}$ , and corresponding Christoffel symbols,  $\Gamma_{\beta\gamma}^\alpha$ , at a given space-time point, according to the IAU2000 conventions. This requires a central planetary potential model (such as GRIM5-S1 given in the International Terrestrial Reference System (ITRS) for the Earth), plus planetary positions and velocities provided in ephemeris, such as DE403, in terms of TDB (“Temps Dynamique Barycentrique”) or TCB (“Temps de Coordonnée Barycentrique” related to the Barycentric Coordinate Reference System (BCRS)). Additional relativistic (hence four-dimensional) space-time transformations are needed to transform between the different relativistic reference frames.